Continuous time one-dimensional asset pricing models with analytic price-dividend functions*

Yu Chen
Department of Mathematics
Idaho State University
Pocatello, ID 83209

Thomas F. Cosimano
Department of Finance
Mendoza College of Business
University of Notre Dame
E-mail: thomas.f.cosimano.1@nd.edu
Phone: (574) 631-5178
Fax: (574) 631-5255

Alex A. Himonas
Department of Mathematics
University of Notre Dame

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Abstract: A continuous time one-dimensional asset pricing model can be described by a second order linear ordinary differential equation which represents equilibrium or a no arbitrage condition within the economy. If the stochastic discount factor and dividend process are analytic, then the resulting differential equation has analytic coefficients. Under these circumstances, the one-dimensional Cauchy-Kovalevsky Theorem can be used to prove that the solution to such an asset pricing model is analytic. Also, this theorem allows for the development of a recursive rule which speeds up the computation of an approximate solution. In addition, this theorem yields a uniform bound on the error in the numerical solution. Thus, the Cauchy-Kovalevsky Theorem yields a quick and accurate solution of many known asset pricing models.

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1 Introduction

Most applied research on asset pricing in continuous time assumes a linear structure for the stochastic discount factor (SDF) or risk free interest rate.[1] Researchers make this assumption since there are closed form solutions for asset prices in this set up. However, it is known from the equity premium literature that non-linear SDF are necessary to capture the dynamic behavior of the equity premium.[2] In this paper we consider such asset pricing models in which the SDF and dividend process are analytic. Using power series methods we show that there solutions are analytic and quickly compute polynomial approximations with precise error estimates.[3]

An one-dimensional asset pricing model in continuous time is characterized by an ordinary differential equation (ODE) whose solution is the price or return of the asset under study. There are only a few examples of such models whose solution can be expressed in closed form. The most general such models are those whose the conditional expected SDF is linear in the state variable and the conditional variance of this SDF is also linear in the state variable. They are called affine models and their solutions are log-linear in the state variable.[4] Observe for these simple asset pricing models the linearity of the ODE coefficients leads to log-linear solutions.

In analogy for general non-affine models analytic characteristics (SDF and dividend process), which translate into analytic coefficients for the ODE of the model, lead to analytic solutions for the price-dividend functions. An analytic asset pricing function $P(x)$, defined on an open interval Ω in $\mathbb{R}$, has the desirable property that it can be represented by a Taylor series in some neighborhood

[1]Constantinides (1990, 1992) is an exception to this rule. Yet, his asset pricing model fits into our class of problems.

[2]Mehra and Prescott (1985). Constantinides (2002), Campbell and Viceira (2002), Mehra and Prescott (2003) and Cochrane (2005, Chapters 20 and 21) provide recent exposition of this work.

[3]Throughout this paper we use CCCH to refer to Calin, Chen, Cosimano and Himonas (2005). In addition, CCH (2006a), and CCH (2006b) for Chen, Cosimano and Himonas (2006a) and (2006b), respectively. These papers show how to use analytic methods to solve discrete time asset pricing models.

[4]See Duffie and Kan (1996), Duffie, Pan, and Singleton (2000), and Dai and Singleton (2000). Cochrane (2005, Chapter 19) demonstrates how affine models are generalizations of earlier work by Vasicek (1977) and Cox, Ingersoll and Ross (1985). Duffie (1996, Chapter 7) and Shreve (2003, Chapter 10) discuss higher dimensional versions of these models which are not dealt with here. The Heath, Jarrow and Morton (1992) model is based on the observed forward rates rather than the SDF. Their model allows for higher polynomial functions but they do not provide a solution. Rather they use numerical methods to solve the problem. Shreve (2003, Chapter 10) shows the relation between Heath, Jarrow and Morton and affine models. More recently Gabaix (2007) develops a linear price-dividend function by engineering the dividend process to cancel any non-linearity in the SDF.

of each $x_0 \in \Omega$.[5] The radius of convergence, r, is the largest interval $(x_0 - r, x_0 + r)$ in which this series converges. The Cauchy-Kovalevsky Theorem states that an ODE with analytic coefficients and given initial data at a point x_0 has a unique analytic solution in a neighborhood of x_0. It is shown here that most applied asset pricing models in one dimension yield an ODE with analytic coefficients, as long as the conditional mean and variance for the equations of motion for the SDF and the state variable are analytic. Thus, the Cauchy-Kovalevsky Theorem applies to most applied asset pricing models in one dimension so that the equilibrium price-dividend function is analytic.

It is shown that the continuous time versions of Mehra and Prescott (1985) and Campbell and Cochrane (1999) yield an ODE with analytic coefficients. Similarly, the continuous time versions of a majority of applied asset pricing models in one-dimensional are described by ODE with analytic coefficients. Therefore, applying the Cauchy-Kovalevsky Theorem we obtain solutions which can be represented as Taylor series about a point, x_0, where the initial data are prescribed. It is important that the one-dimensional Cauchy-Kovalevky Theorem provides an accurate estimate of the radius of convergence, r. Within the interval of convergence, $(x_0 - r, x_0 + r)$, the solution can be approximated to any desired degree of accuracy by a Taylor polynomial approximation.

To illustrate the use of the Cauchy-Kovalevsky Theorem we provide a complete analysis of the numerical solution to the ODE of the Campbell and Cochrane (1999) model. For this ODE we determine the radius of convergence for the Taylor series of its solution from its coefficients. Writing this ODE in its normal form, that is the coefficient, associated with the second derivative, is 1, then the radius of convergence of the solution is the smallest radius of convergence of its 2 coefficients and output function. Here, the size of the radius of convergence, thus obtained, is large enough to include all values of interest to financial economists. The coefficients of the Taylor series are computed recursively by using the power series of the coefficients and the initial data. The numerical solution is a Taylor polynomial approximation of the actual power series solution. The higher the degree of this polynomial the better is the accuracy of the solution and the error is determined by the approximation estimates on the coefficients of the ODE.

The rest of the paper is structured as follows. The next section lays out the analytic method

[5]See CCCH and CCH (2006a) for a discussion of analyticity and how it applies to discrete time asset pricing models.

for solving one-dimensional asset pricing models. Section 3 applies this method to solve the ODE for the price-dividend function in the Mehra-Prescott (1985) model. The fourth section provides a complete analysis of the more complicated Campbell-Cochrane (1999) model. Section five provides the simulation of the Campbell and Cochrane model using the Taylor polynomial approximation method. Final comments are made in the last section.

2 Asset pricing models in one dimension.

In this paper we consider continuous time asset pricing models with one state variable. The representative agent is assumed to choose equity so that the intertemporal Euler condition is

$$0 = \Lambda(t)D(t)dt + E_t\left[d\left[\Lambda(t)P(t)\right]\right].^{6} \tag{2.1}$$

Here, $\Lambda(t)$ is the stochastic discount factor (SDF) for the valuation of an investment, $D(t)$ is the dividend payment from the equity per unit of time, and $P(t)$ is the price of equity. Thus, equation (2.1) reads as follows: *The change in the marginal valuation of dividends and the expected change in the marginal value of stock price sum to zero.*

Asset pricing models are generally distinguished by the following stochastic process for the SDF

$$\frac{d\Lambda}{\Lambda} = \mu_\Lambda(x(t),t)dt + \sigma_\Lambda(x(t),t)d\theta_t, \tag{2.2}$$

where $x(t)$ is the state variable which is assumed to follow an Ito process of the following form.

$$\frac{dx}{x} = \mu_x(x(t),t)dt + \sigma_x(x(t),t)d\omega_t. \tag{2.3}$$

Here $\mu_\Lambda(x(t),t)$ is the instantaneous mean for the SDF, while $\sigma_\Lambda(x(t),t)$ is its standard deviation. Also $\mu_x(x(t),t)$ is the instantaneous mean for the state variable and $\sigma_x(x(t),t)$ is its standard deviation. The stochastic shocks $d\theta_t$ and $d\omega_t$ are assumed to be Brownian motions.[7]

[6] See Duffie (1996), Cochrane (2005) and Campbell and Viceira (2002). Strictly speaking a representative agent is not necessary. The absence of arbitrage opportunities is sufficient for the existence of a positive pricing kernel so that this condition is satisfied.

[7] To conserve on space we limit the discussion of the underlining Brownian motion and Ito's rule, since the focus of the paper is on solving the resulting equation which represent these asset pricing models. Arnold (1993), Duffie (1996) or Shreve (2003) are good sources for the derivation of these differential equations as well as the vast literature on this subject.

In applied asset pricing models the instantaneous means and standard deviations for the stochastic processes for the SDF and the state variable are usually assumed to be analytic functions. Included in this class of SDF are the above mentioned affine models, Epstein and Zin (1989, 1990, 1991), Abel (1990,1999), Constantinides (1990, 1992), Duffie and Epstein (1992a, b) using the Kreps-Porteus (1978) functional form, Campbell and Cochrane (1999), Campbell and Viceira (2002), and Bansal and Yaron (2004).

To illustrate our method in this work we shall analyze in detail two asset pricing models. The first and simplest is the continuous time versions of Mehra-Prescott's (1985) asset pricing model, where the resulting ODE has coefficients and output which are linear in the state variable. The second, which was introduced by Campbell and Cochrane (1999) to explain the equity premium and the variation of volatility over time, results in an ODE with non-linear but analytic coefficients. Since both models have analytic coefficients, there solutions are analytic and therefore can be represented as power series about the point where the initial data are known. In each case the numerical solution is a Taylor polynomial approximation of the power series. Choosing the order of this polynomial high enough, we can achieve any desired degree of accuracy. The same method can be used for accurately solving a vast majority of asset pricing models.

Given the SDF and the stochastic processes for the state variable, Ito's Lemma yields a stochastic process for the price of the financial asset in which

$$\frac{dP}{P} = \mu_P(x, P', P'', t)dt + \sigma_P(x, P', P'', t)d\omega_t, \tag{2.4}$$

where $P(x)$ is the equilibrium price function for the financial asset.[8] Here $\mu_P(x(t), t)$ is the instantaneous mean for the price function, and $\sigma_P(x(t), t)$ is its standard deviations.

Using Ito's Lemma again, it can be seen that this equilibrium price function satisfies a second order linear ODE, which as an initial value problem (IVP) has the following form

$$P''(x) + a(x)P'(x) + b(x)P(x) = g(x), \; P(x_0) = p_0, \; P'(x_0) = p_1, \tag{2.5}$$

where the coefficients $a(x)$, $b(x)$, and the output $g(x)$ are analytic near the point x_0.

[8] For notational purposes stack the stochastic processes for θ and ω together and call it ω.

Recall a smooth function $f(x)$ is analytic near a point x_0 if it can be represented by its Taylor series, that is

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k, \tag{2.6}$$

as long as $|x - x_0| < r$, where r is the radius of convergence.

The solution to the IVP (2.5) is analytic at x_0. This is a special case of the well known Cauchy-Kovalevsky Theorem . While this theorem holds for both linear and non-linear differential equations in one and several variables, here we shall state it for second order linear differential equations of the form (2.5).[9] For simplicity we shall also assume $x_0 = 0$, since otherwise it can be reduced to this case by a simply change of variable (translation).

Theorem 2.1. *The initial value problem (2.5) has a unique solution $P(x)$ near $x_0 = 0$, which is analytic with radius of convergence, r_0, equal to at least the smallest radius of convergence of the coefficients and the output.*

This theorem qualifies the radius of convergence to be "at least" the smallest radius of convergence. To see why we provide the following example.

Example. The solution to the initial value problem

$$y'' - \frac{1}{x-1}y' = 0, \;\; y(0) = \frac{1}{2}, \;\; y'(0) = -1,$$

is given by $y(x) = \frac{1}{2}(x-1)^2$. It is an analytic function with radius of convergence equal to infinity. However, Theorem 2.1 asserts only that its radius of convergence is greater or equal to 1.

The proof of Theorem 2.1 is provided in the appendix. There are two benefits of this proof. First, it points to a procedure for solving the IVP (2.5). This procedure begins with a formal power series expansion for the solution to the IVP of the following form

$$P(x) = \sum_{k=0}^{\infty} p_k x^k, \tag{2.7}$$

where p_k are to be determined. Substituting this together with the known Taylor series for the coefficients and the output into the IVP and manipulating the result using the operational rules for

[9] See Coddington (1961) or Simmons (1991).

power series we obtain a recurrence relation for the coefficients of the solution, p_k. Then assuming that the Taylor series of the coefficients and the output has radius of convergence r (which is taken to be the optimal) and using the recurrence relation we are able to show that the coefficients p_k satisfy appropriate estimates so that the radius of convergence of the power series (2.7) is r. Thus, the formal power series solution (2.7) provides an honest power series solution to the IVP (2.5).

The second benefit of the proof of Theorem 2.1 is that it yields an accurate estimate of the difference between the power series solution (2.7) and its Taylor's polynomial approximation. More precisely, if

$$P_n(x) = \sum_{k=0}^{n} p_k x^k \tag{2.8}$$

is the n^{th} order Taylor polynomial approximation of the power series solution (2.7), then the error is

$$R_n(x) = P(x) - P_n(x) = \sum_{k=n+1}^{\infty} p_k x^k. \tag{2.9}$$

This error, $R_n(x)$, can be estimated in terms of the coefficients $a(x)$ and $b(x)$, the output, $g(x)$, and the initial data p_0 and p_1. For this we write

$$a(x) = \sum_{k=0}^{\infty} a_k x^k, \quad b(x) = \sum_{k=0}^{\infty} b_k x^k, \quad \text{and} \quad g(x) = \sum_{k=0}^{\infty} d_k x^k. \tag{2.9}$$

and choose r such that $0 < r < r_0$, where r_0 is as in Theorem 2.1. Since r is smaller than the radius of convergence $a(x)$, $b(x)$, and $g(x)$, there exists non-negative constants M_a, M_b, and M_g such that

$$|a_k| \leq \frac{M_a}{r^k}, \quad |b_k| \leq \frac{M_b}{r^k}, \quad \text{and} \quad |d_k| \leq \frac{M_g}{r^k}, \quad k = 0, 1, 2, \ldots. \tag{2.10}$$

With this information in mind, the following corollary provides a uniform bound for the error $R_n(x)$.

Corollary 2.2. *The error $R_n(x)$ between the solution $P(x)$ and its n^{th} order Taylor approximation is estimated as follows*

$$|R_n(x)| \leq \frac{1}{2}\left[M_g + |p_1|(1+r)M + |p_0|M\right] \sum_{k=n+1}^{\infty} \prod_{l=2}^{k-1} \left[\frac{l-1}{r(l+1)} + M\frac{l+r}{(l+1)l}\right] (\mu r)^k, \quad |x| < \mu r,$$

where $M = \max\{M_a, M_b\}$ and $0 < \mu < 1$.

Thus, by adding a sufficient number of coefficients to the Taylor polynomial approximation (2.8), we obtain an accurate enough numerical solution.

Once the solution to the price of the financial asset is known, the stochastic process for the price is given by (2.4) where the first and second derivatives of the solution are substituted for P' and P''. In addition, the stochastic process for the return on the financial asset is given by

$$dR = \mu_R(x, P', P'', t)dt + \sigma_R(x, P', P'', t)dz_t, \tag{2.11}$$

so that the solution to the ODE can also be used to calculate the stochastic process for the return on the financial asset.

3 Mehra and Prescott's asset pricing model.

We begin with the Mehra-Prescott model which might be the simplest application of our method. In this model the stochastic discount factor is

$$\Lambda = e^{-\beta t} [C]^{-\gamma}, \tag{3.1}$$

where $\gamma > 0$ is the coefficient of relative risk aversion, β is the investor's discount factor and C is the level of consumption.[10] The growth rate of consumption is assumed to follow the stochastic process

$$\frac{dC}{C} = dx = (\varphi - 1)(x - \bar{x})dt + \sigma d\omega, \tag{3.2}$$

where $|\varphi| < 1$ is a constant, $d\omega$ is a standard Brownian motion, and σ is the instantaneous standard deviation which is a positive constant.[11] In the equilibrium of the model it is assumed that consumption growth and dividend growth are identical.

Following Arnold (1974), the stochastic differential equation (3.2) has the solution given by

$$x(t) - \bar{x} = \exp[(\varphi - 1)t] [x(0) - \bar{x}] + \sigma \int_0^t \exp[(\varphi - 1)(t - s)] d\omega. \tag{3.3}$$

Using the result from Shreve (2004), we have

$$\sigma \int_0^t \exp[(\varphi - 1)(t - s)] d\omega \sim N\left(0, \frac{\sigma^2}{2(1 - \varphi)} [1 - \exp[2(\varphi - 1)t]]\right),$$

[10]The dependence on time t is dropped when it is obvious by the context.

[11]This stochastic process is referred to as the Ohrnstein-Uhlembeck process which is the continuous time version of an AR(1) which was used by CCCH in the discrete time version of the Mehra-Prescott model.

so that the state variable, x, is normally distributed.

The optimal condition for the investor (2.1) with the SDF (3.1), and the stochastic process for the state variable (3.2) yields the following ODE in the price-dividend function, $P(z)$,

$$c_2 P''(x) = c_1(x)P'(x) + c_0(x)P(x) - 1\,, \tag{3.4}$$

after repeated application of Ito's lemma. The coefficients of this ODE are given by

$$c_2 = \frac{1}{2}\sigma^2,\; c_1(x) = (\gamma-1)\sigma^2 + (1-\varphi)x,\; \text{and } c_0(x) = \beta - \frac{1}{2}(\gamma-1)\gamma\sigma^2 + (\gamma-1)(\varphi-1)x.$$

The subscript j, where $j = 0, 1, 2$, refers to the order of the derivative for the price-dividend function associated with the coefficient.[12]

Dividing equation (3.4) by c_2, and writing it in its normal form, we see that its coefficients and output are affine functions (first degree polynomials), and therefore analytic with infinite radius of convergence. As a result, applying Theorem 2.1 yields that the price-dividend function $P(x)$ is also analytic with infinite radius of convergence. Therefore, it is a power series given by

$$P(x) = \sum_{j=0}^{\infty} p_j x^j. \tag{3.5}$$

Substituting this power series and its first two derivatives into the ODE (3.4), and equating the coefficients associated with x^j yield the following recursive rule for determining the coefficients, p_j, for the power series (3.5)

$$\begin{cases} p_2 = (\gamma-1)p_1 + \frac{1}{2}k_0 p_0\,, \\ p_{j+2} = \frac{1}{(j+1)(j+2)}\left[(j+1)k_3 p_{j+1} + (k_2 j + k_1)c_j + k_0 c_{j-1}\right] \qquad \text{for } j > 1\,, \end{cases} \tag{3.6}$$

where the constants k_0, k_1, and k_2 are defined as

$$k_0 = \frac{2(\gamma-1)(\varphi-1)}{\sigma^2}, k_1 = \frac{2\beta}{\sigma^2} - (\gamma-1)\gamma, k_2 = \frac{2(1-\gamma)}{\sigma^2},\; \text{and } k_3 = 2(\gamma-1).$$

The coefficients p_j defined by the recurrence formula (3.6) are completely determined if we specify the initial conditions $P(0) = p_0$ and $P'(0) = p_1$. Using the average value of the price dividend function on a monthly basis, which is found in the last column of Table 1 (see Wachter

[12] Recall that throughout the paper the state variable is translated by its steady state value $\bar{x}$.

(2002)), we set $p_0 = 30.92 \times 12$. We determine the second initial condition using the historic average equity premium. We begin with the risk free interest rate given by

$$R^b(x) = -E_t\left[\frac{d\Lambda}{\Lambda}\right] = \beta + \gamma(\varphi - 1)(x - \bar{x}) - \frac{\sigma^2}{2}\gamma(\gamma + 1), \tag{3.7}$$

where $E_t[f(x)]$ is the expectation of $f(x)$ conditional on information at time t. Next, the price-dividend function can be used to find the stochastic process for stock returns (2.11), which is given by

$$dR^e(x) = E_t\left[R^e(x)\right] dt + \Sigma(x) d\omega, \tag{3.8}$$

where the instantaneous expected return on stocks is

$$E_t\left[R^e(x)\right] = \frac{\left[1 + P'(x)\left[(\varphi - 1)(x - \bar{x}) + \sigma^2\right] + \frac{1}{2}P''(x)\sigma^2\right]}{P(x)} + (\varphi - 1)(x - \bar{x}), \tag{3.9}$$

and its instantaneous standard deviation is

$$\Sigma(x) = \left[\frac{P'(x)}{P(x)} + 1\right]\sigma. \tag{3.10}$$

Manipulating equations (3.4), (3.7), (3.9), and solving for $P'(z)$ gives

$$P'(x) = \left[\frac{E_t\left[R^e(x)\right] - R^b(x)}{\gamma\sigma^2} - 1\right] P(x). \tag{3.11}$$

Using equation (3.11) with an equity premium of 0.049/12, and appropriate values of the parameters we compute the second initial condition $P'(0)$. For an example, following Wachter (2002), if $\gamma = 1.1$, $\sigma = 0.00363$, $\varphi = 0.9851$, $\bar{x} = 0.001633$, and $\beta = 0.00163$, then $p_1 = P'(0) = 103,736$. However, this value is too large. As a consequence, the price-dividend function is unrealistic since it would be negative for consumption growth of only $-\sigma$. This conclusion is just another reflection of the equity premium puzzle. To see this, we combine (3.11) together with its standard deviation of stock returns (3.10) to find that

$$\gamma\sigma = \frac{E_t\left[R^e(x)\right] - R^b(x)}{\Sigma(x)}. \tag{3.12}$$

This is the familiar relation between the standard deviation of consumption growth, σ, and the Sharpe ratio, given by the right hand side of (3.12).[13] Now a low coefficient of relative risk aversion

[13]See Cochrane (2005, p. 456 equation (21.2)).

$\gamma = 1.1$ multiplied by a low standard deviation of consumption growth $\gamma\sigma = 0.0040$ imply from (3.12), that a high equity premium $E_t\left[R^e(x)\right] - R^b(x)$ must be matched by an even higher standard deviation of stock returns, $\Sigma(x)$. Thus, $P'(0)$ needs to be significantly higher than $P(0)$. By lowering the equity premium to the value found in CCCH, 0.006/12, the second initial condition is cut by a factor of ten to $p_1 = 12,376$, since the standard deviation of stock returns does not have to be as high. In this case, the analytic price-dividend function in Figure 1 is well behaved for $z \in [-6\sigma, 6\sigma]$.[14] Thus, the equity premium puzzle is still contained in the continuous time version of the Mehra-Prescott model in that the low standard deviation of consumption growth cannot be reconciled with the equity premium without a significant increase in the coefficient of relative risk aversion.

The numerical solution for the Mehra and Prescott model is simply a Taylor polynomial approximation of the power series (3.5), that is we replace ∞, the upper limit, with a large number n. The error is estimated by using Corollary 2.2. An 85^{th} order Taylor polynomial leads to an error less than 10^{-16}. Figure 1 displays the Taylor polynomial approximation for the price-dividend function in this case.

4 Campbell and Cochrane's asset pricing model.

Next, we consider Campbell and Cochrane's (1999) asset pricing model, which has non-linear but analytic coefficients near zero. This model's SDF is designed to capture the time variation in equity premium observed in the historical data.[15] It depends on the consumption of the investor, $C(t)$, and the surplus consumption ratio, $S(t) = \frac{C(t)-X(t)}{C(t)}$, which measures how close consumption is to past habits, $X(t)$. More precisely, it is of the following form

$$\Lambda = e^{-\beta t}\left[SC\right]^{-\gamma}. \tag{4.1}$$

Following Campbell and Cochrane, we use the new variables defined by

$$C = e^x, \quad \text{and} \quad S = e^s,$$

[14] The solution is found using a 100 order Taylor approximation of the price-dividend function using the same procedure described in the Campbell and Cochrane model below.

[15] Cochrane (2005 Chapter 20) provides a recent analysis of the empirical facts while Chapter 21 explains how the Campbell and Cochrane model captures these concepts.

so that both consumption and the surplus consumption ratio are always positive. Then, the consumption growth, dx, is assumed to be a random walk with drift $\bar{x}$ of the form

$$dx = \bar{x}dt + \sigma d\omega, \tag{4.2}$$

where the random shock to consumption growth, $d\omega$, is a standard Brownian motion. Consequently, consumption growth is not a state variable for the price-dividend function. The only state variable in the model is the surplus consumption ratio which follows the stochastic process

$$ds = (\phi - 1)(s - \bar{s})dt + \lambda(s - \bar{s})\sigma d\omega, \tag{4.3}$$

where $\bar{s}$ is the logarithm of the steady state surplus consumption ratio and the sensitivity function $\lambda(s - \bar{s})$ is defined by

$$\lambda(s - \bar{s}) = \begin{cases} \frac{1}{\bar{S}}\sqrt{1 - 2(s - \bar{s})} - 1 & \text{if } s < \bar{s} + \frac{1-(\bar{S})^2}{2}, \\ 0 & \text{if } s \geqslant \bar{s} + \frac{1-(\bar{S})^2}{2}, \end{cases} \tag{4.4}$$

where

$$\bar{S} = \sigma\sqrt{\frac{\phi\gamma}{1 - \phi - \frac{b}{\gamma}}}.$$ [16]

This sensitivity function is designed to increase the standard deviation of the surplus consumption ratio by multiplying the random shocks to consumption growth $\sigma d\omega$. Also, it is chosen so that the investor's habits are only dependent on the consumption level of others. Furthermore, it assures that random shocks are magnified during bad times and minimized during prosperous times.[17] Finally, the sensitivity function leads to a risk free rate which is a linear function of the surplus consumption ratio.

To show the linearity of the risk free interest rate we begin with the stochastic process for the SDF in Campbell and Cochrane's model,

$$\frac{d\Lambda}{\Lambda} = \left[\gamma(1 - \phi)s - \beta - \gamma\bar{x} + \frac{\gamma^2\sigma^2}{2}\left(1 + \lambda(s)\right)^2\right] dt - \gamma\sigma\left(1 + \lambda(s)\right)^2 d\omega\,, \tag{4.5}$$

which is a specific functional form of (2.2). Note that in (4.5) s stands for $s - \bar{s}$. Also, observe that the instantaneous mean and standard deviation of (4.5) are analytic whenever the sensitivity

[16] In the Campbell and Cochrane paper they set $b = 0$, while $b \neq 0$ in the Wachter (2002, 2006) models.

[17] See CCH (2006a) for simulations of this stochastic process in discrete time. The steady state distribution of the surplus consumption ratio is derived below.

function $\lambda(s)$ is analytic. Following Cochrane (2005, p. 29), we use the basic pricing relation (2.1) together with (4.5) and the definition of $\lambda(s)$ to obtain

$$R^b(s) = -E_t\left[\frac{d\Lambda}{\Lambda}\right] = \gamma(\phi-1)s + \beta + \gamma\bar{x} - \frac{\gamma^2\sigma^2}{2}\left(1+\lambda(s)\right)^2 = r^b - bs\,, \tag{4.6}$$

where

$$r^b = \beta + \gamma\bar{x} - \frac{1}{2}\left(\gamma(1-\phi) - b\right).$$

Thus, the risk free interest rate is a linear function of the surplus consumption ratio.

Substituting the stochastic processes for the SDF (4.5), consumption growth (4.2), and the surplus consumption ratio (4.3) into (2.1), and applying Ito's Lemma leads to the following second order linear ODE for the price-dividend function, $P(s)$,

$$c_2(s)P''(s) = c_1(s)P'(s) + c_0(s)P(s) - 1\,,^{18} \tag{4.7}$$

where

$$c_2(s) = \frac{\sigma^2(1+\bar{S}^2)}{2\bar{S}^2} - \frac{\sigma^2}{\bar{S}^2}s - \frac{\sigma^2}{\bar{S}}r(s),$$

$$c_1(s) = \frac{\sigma^2(\bar{S}^2+\gamma)}{\bar{S}^2} + \frac{K_1\bar{S}^2 - 2\gamma\sigma^2}{\bar{S}^2}s - \frac{\sigma^2(1+\gamma)}{\bar{S}}r(s),$$

and

$$c_0(s) = \frac{2K_0\bar{S}^2 - \sigma^2\gamma^2 - \sigma^2\bar{S}^2}{2\bar{S}^2} + \frac{\sigma^2\gamma^2 - \gamma K_1\bar{S}^2}{\bar{S}^2}s + \frac{\sigma^2\gamma}{\bar{S}}r(s).$$

Here, $K_0 = \beta + (\gamma-1)\bar{z} > 0$, $K_1 = (1-\phi) > 0$, and

$$r(s) \doteq \bar{S}\Big(\lambda(s)+1\Big) = \begin{cases} \sqrt{1-2s} & \text{if } s < \frac{1-\bar{S}^2}{2}, \\ \bar{S} & \text{if } s \geq \frac{1-\bar{S}^2}{2}. \end{cases}$$

The normal form of equation (4.7) is

$$P''(s) + a(s)P'(s) + b(s)P(s) = g(s)\,, \tag{4.8}$$

where

$$a(s) = -\frac{c_1(s)}{c_2(s}, b(s) = -\frac{c_0(s)}{c_2(s)} \text{ and } g(s) = -\frac{1}{c_2(s)}. \tag{4.9}$$

[18] Wachter (2005) derives this ODE for the Campbell and Cochrane model using no arbitrage techniques as in Duffie (Chapter 6 and 10) rather than equilibrium arguments as in Lucas (1978), which is used here.

To apply the Cauchy-Kovalevsky Theorem for equation (4.8) we need to determine the radius of convergence of the coefficients and the output. Looking at their definition, we see we must impose the two following conditions. First, $c_2(s)$ must be positive, which is true when

$$|s| < \frac{1-\bar{S}^2}{2}.$$

Second, $r(s)$ must be analytic. Since $\sqrt{1-2s}$ is analytic, whenever $1-2s>0$, we see that $r(s)$ is analytic, if $s<\frac{1}{2}$. Since the nearest to $s=0$ singularity of $r(s)$ in the complex plain is the point $s=\frac{1}{2}$, the radius of convergence of the power series of $r(x)$ about $s=0$ is equal to the distance from 0 to $\frac{1}{2}$. That is the optimal radius of convergence for $r(s)$ is $\frac{1}{2}$. Since $r(s)$ and $\frac{1}{r(s)}$ have the same radius of convergence about 0, we conclude that the radius of convergence of the coefficients $a(s)$, $b(s)$, and the output $g(s)$ is the smallest to the smallest of the numbers $\frac{1}{2}$ and $\frac{1-\bar{S}^2}{2}$. However, $\frac{1-\bar{S}^2}{2}<\frac{1}{2}$. Therefore, the radius of convergence r_0 of the coefficients $a(s)$, $b(s)$, and the output $g(s)$ is

$$r_0 = \frac{1-\bar{S}^2}{2}. \tag{4.10}$$

Finally, applying Theorem 2.1, we conclude that the solution for the price-dividend function $P(s)$ of the Campbell and Cochrane model is analytic near zero and its Taylor series

$$P(s) = \sum_{j=0}^{\infty} p_j s^j, \tag{4.11}$$

has radius of convergence r_0 given by (4.10).

Recursive rule for coefficients of power series. Next, we derive the recurrence relation for determining the coefficients p_j. For this, we need to write the Taylor series for the functions $c_0(s)$, $c_1(s)$ and $c_2(s)$. Observe that each of these coefficients has the functional form

$$c(a_0,a_1,a_2,s) = a_0 + a_1 s + a_2 r(s), \tag{4.12}$$

for some $(a_0,a_1,a_2)\in\mathbb{R}^3$, so that the derivatives of these coefficients are dependent on the derivatives of $r(s)$. These derivatives are given by

$$r^{(n)}(s) = \begin{cases} r(s) & \text{if } n = 0, \\ -\frac{1}{1-2s} r(s) & \text{if } n = 1, \\ -\frac{(2n-3)!!}{(1-2s)^n} r(s) & \text{if } n \geqslant 2 \end{cases} \quad \text{and} \quad r^{(n)}(0) = \begin{cases} 1 & \text{if } n = 0, \\ -1 & \text{if } n = 1, \\ -(2n-3)!! & \text{if } n \geqslant 2. \end{cases}$$ [19]

Therefore, the derivatives of any of the coefficients $c(a_0, a_1, a_2, s)$ are

$$c^{(n)}(a_0, a_1, a_2; s) = \begin{cases} a_0 + a_1 s + a_2 r(s) & \text{if } n = 0, \\ a_1 + a_2 r^{(1)}(s) & \text{if } n = 1, \\ a_2 r^{(n)}(s) & \text{if } n \geqslant 2. \end{cases}$$

As a result,

$$c^{(n)}(a_0, a_1, a_2; 0) = \begin{cases} a_0 + a_2 & \text{if } n = 0, \\ a_1 - a_2 & \text{if } n = 1, \\ -a_2(2n-3)!! & \text{if } n \geqslant 2. \end{cases}$$

So

$$c^{(n)}(a_0, a_1, a_2) = \begin{cases} a_0 + a_2 & \text{if } n = 0, \\ a_1 - a_2 & \text{if } n = 1, \\ -\frac{a_2(2n-3)!!}{n!} & \text{if } n \geqslant 2. \end{cases}$$

Here the abbreviation $c_j^{(n)} = \frac{1}{n!} c_j^{(n)}(0)$ is used for $j = 0, 1, 2$ and $n = 0, 1, 2, \dots$.

Substituting into equation (4.7) the power series (4.11) for $P(s)$, its first two derivatives and the power series for the coefficients $c_0(s)$, $c_1(s)$, and $c_2(s)$ gives the following recurrence relations

$$\begin{aligned} 2c_2^{(0)} p_2 &= c_1^{(0)} p_1 + c_0^{(0)} p_0 - 1, \text{ and} \\ (j+1)(j+2) c_2^{(0)} p_{j+2} &= \sum_{k=2}^{j} \left[c_0^{(j-k)} + k c_1^{(j-k+1)} - (k-1) k c_2^{(j-k+2)} \right] p_k \\ &\quad + (j+1)(c_1^{(0)} - j c_2^{(1)}) p_{j+1} + (c_0^{(j-1)} + c_1^{(j)}) p_1 + c_0^{(n)} p_0 \,. \end{aligned} \tag{4.13}$$

Initial conditions. To determine p_j recursively from the formulas (4.13) we need to know the initial conditions $P(0) = p_0$ and $P'(0) = p_1$. The first initial condition is chosen to be $p_0 = 219.60$, so that the price-dividend ratio would be the same as in Campbell and Cochrane (1999).

To choose the second initial condition we follow the same strategy as in the Mehra-Prescott model. In the Campbell and Cochrane model the return on equity (2.11) is given

$$dR^e(s) = E_t \left[R^e(s) \right] dt + \Sigma(s) \, d\omega \,, \tag{4.14}$$

[19]The super script $^{(n)}$ refers to the n^{th} order derivative. The notation !! means $7!! = 7 \cdot 5 \cdot 3 \cdot 1$.

where the instantaneous expected return on equity is

$$E_t\left[R^e(s)\right] = \bar{x} + \frac{1}{2}\sigma^2 + \frac{(\phi-1)sP'(s) + \frac{\sigma^2}{2}\lambda(s)^2P''(s) + \sigma^2\lambda(s)P'(s) + 1}{P(s)}, \tag{4.15}$$

and the instantaneous standard deviation for the return on equity is

$$\Sigma(s) = \left(\frac{\lambda(s)P'(s)}{P(s)} + 1\right)\sigma. \tag{4.16}$$

In addition, the risk free return on bonds is given by equation (4.6). Consequently, the Sharpe ratio is given by

$$S(s) = \frac{E_t\left[R^e(s)\right] - \left[r^b + bs\right]}{\Sigma(s)}. \tag{4.17}$$

Substituting the instantaneous return on stocks and bonds into the ODE (4.7) we obtain

$$P'(s) = \frac{\{E_t[R^e(s)] - R^b(s) - \sigma^2\gamma(1+\lambda(s))\}P(s)}{\gamma\sigma^2\lambda(s)(1+\lambda(s))}. \tag{4.18}$$

Then evaluating (4.18) at $s = 0$ determines the second initial condition

$$p_1 = P'(0) = \frac{\left\{E_t[R^e(0)] - r^b - \frac{\gamma\sigma^2}{S}\right\}p_0}{\frac{\gamma\sigma^2}{S}\left(\frac{1}{S} - 1\right)}. \tag{4.19}$$

The value of p_1 is found by replacing $E_t\left[R^e(0)\right] - r^b$ with the average equity premium. In the simulation this initial condition is used to set $p_1 = 230.00$.

Thus, the average price-dividend ratio and equity premium in the economy are used to establish the necessary conditions for the Cauchy-Kovalevsky Theorem 2.1 to hold. Consequently, the equilibrium price-dividend ratio for the Campbell and Cochrane model is the Taylor series around $s = 0$ with radius of convergence at least equal to $r = 0.4970$. In addition, the instantaneous mean and standard deviation for stock returns, given by (4.15) and (4.16), are analytic within the same interval of convergence.

This condition is akin to the condition for the state-price beta model in the consumption CAPM developed by Duffie (1996, pp. 101-108 and pp. 227-230). This condition also satisfies the no arbitrage condition between stocks and bonds.[20] As in the Mehra and Prescott model the no

[20] Wachter (2005) derives the continuous time ODE for the Campbell and Cochrane model by starting with this no arbitrage condition rather than the equilibrium approach used here.

arbitrage condition (4.18), the standard deviation (4.16) and Sharpe ratio (4.17) can be combine to yield

$$S(s) = \gamma\sigma(1 + \lambda(s))$$

so that the equity premium puzzle can be resolved in the Campbell and Cochrane model through the increased sensitivity of the random shock to consumption growth on the surplus consumption ratio. In particular, $\lambda(0) + 1 = \frac{1}{\bar{S}} = 12.92$ for the parameter values used in the simulation of the Campbell and Cochrane model so that the Sharpe ratio, $\gamma\sigma(1+\lambda(s))$, is close to its historic average.

Numerical solution and error analysis. The numerical solution of Campbell and Cochrane's model (4.7) is a n^{th} degree Taylor polynomial approximation, $P_n(s)$, to the power series expansion (4.11) of the price-dividend function, that is

$$P_n(s) = \sum_{j=0}^{n} p_j s^j. \tag{4.20}$$

The bigger the n the more accurate is the numerical solution $P_n(s)$. To estimate the error $P(s)-P_n(s)$ we apply Corollary 2.2. However, to apply Corollary 2.2 we need to establish a uniform bound on the coefficients, $a(s) = -c_1(s)/c_2(s)$, $b(s) = -c_0(s)/c_2(s)$, and the output, $g(s) = -1/c_2(s)$ on a circle centered at 0, and of radius r in the complex plane. Note that $c_2(s) \to 0$ as $s \to r_0 = \frac{1-\bar{S}^2}{2} = 0.4970$. For this, we choose r smaller r_0, say $r = 0.4$, and restrict the domain of definition for the coefficients and the output to $|z| \leq r$. Then, using the Cauchy integral formula we compute the constants M_a, M_b, and M_g used in the estimates (2.10). For example, if $r = 0.4$, then $M \doteq min\{M_a, M_b\} = 112.7642$,and $M_g = 4187.0441$.

Applying Corollary 2.2 with these values of M and M_g we find a uniform bound on the Taylor series remainder (numerical solution error) $P(s) - P_n(s)$. For $\mu = 0.5$ and $n = 179$ this error is less than 10^{-9}, while for $\mu = 0.8$ the degree of the Taylor polynomial approximation must be increased to $n = 813$ to obtain the same degree of accuracy.[21] Thus, if we want the support of the distribution of the surplus consumption ratio to be $S \doteq e^s \in [0.06335, 0.09450]$, then we must choose the Taylor

[21] The formula in Corollary 2.2 contains a sum to ∞, however the computer cannot count this high. Consequently, we compared the error when the number of terms was 1500 and 3000. The change in error was only 1.2372×10^{-79}, so that this source of error is not significant enough to change the error bound at the level of accuracy of 10^{-16}.

polynomial approximation of degree greater than or equal to 179^{th} in order to keep the error do to the Taylor remainder less than 10^{-9}. However, if the support is increased to $S \in [0.0032, 0.1066]$, then the degree of the Taylor polynomial approximation must increase to 813^{th} to maintain the same accuracy for the price-dividend ratio.[22] Using a standard PC and Maple, the 179^{th} degree polynomial approximation of the solution, as well as all the graphs related to the numerical solution in this paper are calculated in 10 seconds, while it takes 90 seconds for the 813^{th} order polynomial. Thus, the analytic method produces an accurate solution to Campbell and Cochrane's model in minimal time.

Stationary distribution of surplus consumption ratio. The price-dividend function in the Campbell and Cochrane model is analytic for $|s| \leq r$. Consequently, we want to restrict the steady state probability distribution for the surplus consumption ratio to a support that is a closed subset of the interval $[-r, r]$. This restriction assures that the price-dividend function is analytic for every possible realization of the surplus consumption ratio. Note in the original Campbell and Cochrane working paper this support was chosen to be [???] rather than $[-r, r]$, since they did not determine the radius of convergence for the price-dividend solution.

Merton (1990, Chapter 17), and Cox and Miller (1965) provide the mathematical argument for determining the probability distribution of a random variable which follows a stochastic process of the form

$$ds = b(s)dt + [a(s)]^{\frac{1}{2}} d\omega. \tag{4.21}$$

We want to find the stationary probability distribution of s in the Campbell and Cochrane case, that is when $b(s) = (\phi - 1)s$, and $a(s) = \lambda(s)^2\sigma^2$.

For this, let $L(s, t, s_0)$ be the conditional probability density for s at time t given initial s_0. This density function satisfies the Kolmogorov-Fokker-Planck forward equation

$$\frac{1}{2}\frac{\partial^2}{\partial s^2}\left[a(s)L(s,t,s_0)\right] - \frac{\partial}{\partial s}\left[b(s)L(s,t,s_0)\right] = \frac{\partial L}{\partial t}(s,t,s_0).$$ [23]

[22] One concern with such a high order Taylor polynomial is rounding error, since as n increases the coefficients get larger as x^n gets smaller. However, the approximations are not materially effected by this issue. For example, the sup-norm of $P_{100}(s) - P_{50}(s)$ for $|s| \leq .32$ is less than 10^{-9}, so that the solution is already accurate at a 50^{th} order Taylor polynomial approximation for all the circumstances considered in this paper.

[23] See Cox and Miller (1965, pp.208-209).

This equation measures the chance of a small change in s at any instant of time. It is derived by calculating the probability of a small change in s in a small change of time t, using a second order Taylor approximation for this change. Since we are interested in the steady state distribution we want the property

$$\lim_{t\to\infty} L(s,t,s_0) = \pi(s) \text{ so that } \lim_{t\to\infty} \frac{\partial L}{\partial t}(s,t,s_0) = 0.$$

As a result, the steady state distribution for s solves the second order ODE equation

$$\frac{1}{2}\frac{d^2}{ds^2}\left[a(s)\pi(s)\right] - \frac{d}{ds}\left[b(s)\pi(s)\right] = 0. \tag{4.22}$$

In the appendix this ODE is solved subject to a reflection boundary at $0 < s^* \leq r$, so that $\lambda(s) > 0$ for all $|s| \leq s^*$. As a result, the steady state distribution for s is

$$\pi(s) \quad = K \exp\left\{-\tfrac{2\sigma^2 k^4 + (1-\phi)(3-k^2)}{\sigma^2 k^4} \ln \lambda(s) - \tfrac{(1-\phi)}{\sigma^2 k^4}\left[\tfrac{k^2-1}{\lambda(s)} + 3\lambda(s) + \tfrac{\lambda(s)^2}{2}\right]\right\} \tag{4.23}$$

for $s \in [-s^*, s^*]$ and zero otherwise. Here $K = \left[\int_{-s^*}^{s^*} \pi(v)dv\right]^{-1}$ and $k \doteq 1/\bar{S}$.

For the parameter values in the simulation of the Campbell and Cochrane model Figure 2 plots this stationary probability distribution over the support $[-0.8r, 0.8r] = [-0.32, 0.32]$ for s which is skewed to the right.[24] Thus, Theorem 2.1 can be applied to Campbell and Cochrane's ODE (4.7) for all possible realizations of the surplus consumption ratio, since the coefficients and output of this ODE are always analytic under the steady state distribution for the surplus consumption ratio.

5 Simulation of Campbell and Cochrane model

After setting the initial conditions and the support of the distribution of the surplus consumption ratio, the solution of the ODE (4.7) for the Campbell and Cochrane model is unique and analytic over the entire support of the steady state probability distribution for the surplus consumption ratio. For concreteness let this support be $[-\mu r, \mu r]$. In the simulations the parameters are set using a monthly time frame which CCH (2006a) found to best represent the discrete time model. The results of the simulations in Table 1 and Figures 2 − 6 are annualized. The parameters on a

[24]This result contrasts with the simulations of the discrete time model in CCH (2006a). They found a tendency for the distribution of the surplus consumption ratio to be skewed to the left. This graph corresponds to Figure 2 of Campbell and Cochrane (1999) although here the support for the distribution is smaller relative to theirs.

monthly basis are $r^b = 0.00078$, $\bar{x} = 0.00157$, $\phi = 0.9940$, $\gamma = 3.457$, $\sigma = 0.00323$, $b = 0$, $\bar{S} = 0.0774$ and $\mu r = 0.32$.[25] The first initial condition is based on the historic average price-dividend ratio, $p_0 = 219.6$. The second initial condition was tied to the historic average equity premium, following equation (4.19), so that $p_1 = 230.00$.

Table 1 records in column 2 the moments from the solution of Campbell and Cochrane's model. Column 3 records the sample data from Campbell and Cochrane (1999) which is based on the U.S. stock market from 1947 to 1995. Following Campbell and Cochrane, the price-dividend ratio is 18.3 by construction, however, p_0 could be set so that it equals its historic value in column 4 to match the Wachter (2002) data. The price-dividend function in Figure 3 varies from 11.7 to 27.3 as the surplus consumption ratio varies in the interval $[\bar{S}e^{-\mu r}, \bar{S}e^{\mu r}] = [0.0561, 0.1066]$. Thus, there could be a change in the price-dividend function of more than 100% over the support of the steady state distribution of the surplus consumption ratio.

The graph in Figure 3 corresponds to Figure 3 of Campbell and Cochrane. The main difference from their graph is that the price-dividend function is portrayed over a smaller range. We choose the smaller range based on the error analysis. To increase the upper bound of the support to 0.1140, we would have to set $\mu = 0.9$. To reduce the error to 10^{-9} in this case, we would have to increase the order of the Taylor polynomial to $n = 2070$, which cannot be calculated on a standard PC since there is too much rounding error. If we limit the Taylor polynomial to 179^{th} order the error is close to zero in the interval $S \in [0.06335, 0.09450]$. By increasing μ to 0.8 so that $S \in [0.0561, 0.1066]$, the Taylor polynomial must increase to 813^{th} order to keep error the same.[26] Thus, we cannot identify the behavior of the price-dividend ratio over as large a range considered in Campbell and Cochrane (1999), however dividend growth of 32% per month is larger than any historic observation in the Campbell and Cochrane (1999) and Wachter (2002) data sets.

To see the effect of additional coefficients compare the 813^{th} order Taylor polynomial for the price-dividend ratio relative to its first order Taylor polynomial. The $\diamond$ line in Figure 4 shows that

[25]The time frame only materially effects the level of the price-dividend ratio. We tried the original parameters of Campbell and Cochrane but the return on equity was too high.

[26]By the way the price-dividend function becomes unstable for s within 0.000001 of r, so that the graphs in the paper are a good representation of the range of the surplus consumption ratio in which the analytic method can be used.

this error is small when the surplus consumption ratio is close to its steady state value of $\bar{s}$. However, the error is around 3.5% for high surplus consumption ratio and 6.6% for low surplus consumption ratio. By moving to the fourth order Taylor polynomial for the price-dividend ratio, the solid line is close to zero for almost all surplus consumption ratios but can still be close to a 0.6% error for high surplus consumption ratios. This again is a reflection of the non-linear property of the true price-dividend function. By moving to the 813^{th} order Taylor polynomial the change in the solution cannot be detected by the computer.[27]

The conditional expected return on equity given by (4.15) can also be calculated once the price-dividend function is known. The expected return on equity at $S = \bar{S}$ is 8.4% in Table 1. This value of the conditional expected return is close to the value in Campbell and Cochrane's data set. By manipulating the parameter p_1 one can match the expected return on equity exactly. In Figure 5 the expected return on equity, given by the bottom line, changes from 13.9% to 1.4% over the possible range of the surplus consumption ratio. This graph corresponds to Figure 4 of Campbell and Cochrane except that the return declines faster for high surplus consumption ratios. This helps explain the ability of the price-dividend ratio to forecast future returns as demonstrated by Cochrane (2005, 2006). When the price-dividend ratio is above the normal value expected by individuals, the price-dividend ratio moves back toward normal times, so that expected returns are low during these time periods. These lower expected returns lead to lower realized returns as well, following (4.14). Thus, the solution captures the time variation in expected returns envisioned by Campbell and Cochrane.

The conditional standard deviation of stock returns is given by (4.16) for various values of the surplus consumption ratio. This standard deviation is about 15% in Table 1 and the thick line in Figure 5 varies between 7.5% and 20.4% as the surplus consumption ratio varies from 0.1066 to 0.0561 . This result corresponds to Figure 5 of Campbell and Cochrane (1999) for most values of the surplus consumption ratio. However, the decline in the standard deviation at higher levels of the surplus consumption ratio is faster for the true price-dividend function. In Campbell and Cochrane's

[27] This conclusion suggest that linear generated asset pricing model of Gabaix (2007), which leads to a linear price-dividend function, is inconsistent with non-linear asset pricing models such as Campbell and Cochrane's. This inconsistency becomes more pronounced as dividend growth moves further away from its steady state value.

model the volatility of stocks is lowest in good times while it is highest in bad times. This result is consistent with the direction of change in volatility of the stock market over time in that it is lower during expansions.[28]

Finally, the conditional Sharpe ratio can be calculated using (4.17). At the steady state surplus consumption ratio this Sharpe ratio is 0.56 in Table 1, which is close to the historic average found in Campbell and Cochrane's data set. Following the behavior of the mean and standard deviation of equity, the Sharpe ratio in Figure 6 varies between 0.72 and 0.39 as the economy moves from bad to good times. This corresponds to Figure 6 of Campbell and Cochrane (1999).

In experimenting with the parameters the return on equity and the Sharpe ratio moved closer to the historic average by raising the persistence of the surplus consumption ratio by a small amount to $\rho = 0.9949$. In addition, by raising the coefficient of risk aversion a little to $\gamma = 3.6$ the return on equity becomes more aligned with the historic behavior. The more systematic simulated method of moments of Christensen and Kiefer (2000) can be used to choose the optimal combination of parameters for the theory to match the data, since the Maple program takes a few seconds to solve for 225 coefficients in the Taylor polynomial for the price-dividend function.

6 Conclusion

Rather than summarizing the paper, which was done in section 2, we conclude by mentioning that the general Cauchy-Kovalevsky Theorem is applicable to many continuous time problems in finance. In finance, it is customary for continuous time problems, including option pricing, term structure, portfolio decisions, corporate finance, market microstructure and financial engineering, to have SDF which are analytic.[29] Thus, each of these problems can potentially benefit from using the analytic method discussed here. However, some of these problems have several state variables. In future work we plan to extend to multiple dimensions the use of the Cauchy-Kovalevsky Theorem to solve asset pricing models.

[28] See Schwert (1989,1990).

[29] See Sundaresan (2000) for a recent survey of the work in continuous time finance.

Table 1. Comparison of Model Relative to Data

Statistic	Campbell Cochrane	Campbell Cochrane Data	Wachter Data
$E_t(R^e)$	0.084	0.076	0.069
$\sigma(R)$	0.151	0.157	0.163
$E_t(R^b)$	0.009	0.009	0.020
$E_t(R^e - R^b)$	0.073	0.067	0.049
$Sharpe$	0.56	0.34	0.30
P	18.3	24.7	30.92

$Notes:$ R^e is the real return on stocks and R^b is the real return on bonds, and P is the price-dividend ratio. E_t is the conditional expectation operator and σ is the standard deviation. The statistics for the theoretical solutions are evaluated at the historic average for the state variables. The parameters for Campbell and Cochrane's model are $r^b = 0.00078$, $\bar{x} = 0.00157$, $\phi = 0.9940$, $\gamma = 3.457$, $\sigma = 0.00323$, $b = 0$, $p_0 = 219.60$, $p_1 = 230.00$, $\bar{S} = 0.0774$ and $\mu r = 0.4$. The data for Campbell and Cochrane is taken from their Table 4. We use the Postwar Sample from 1947 to 1995 for the U. S.. The Wachter data comes from Table 4 of her 2002 working paper. The sample is quarterly data for the U. S. from 1957 to 1998.

In Figure 1, we show the price-dividend function in the Mehra and Prescott model. The equity premium is set at 0.006/12 so that $p_1 = 12376.84702$. The parameters for the Mehra and Prescott model are $r^b = 0.00016$, $\bar{x} = 0.00163$, $\gamma = 1.1$, $\sigma = 0.00289$, $\sigma_2 = 0.00075$, and $\varphi = 0.9851$. The x-axis allows consumption growth to vary over $x \in [-6\sigma, 6\sigma]$. The y-axis records the price-dividend function.

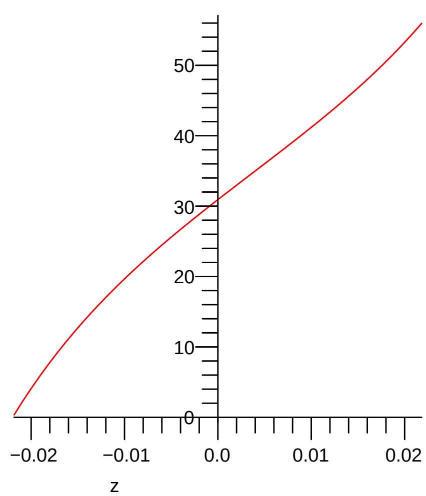

Figure 1

Figure 2 shows the steady state probability distribution in the Campbell and Cochrane model. The parameter values are $r^b = 0.00078$, $\bar{x} = 0.00157$, $\phi = 0.9940$, $\gamma = 3.457$, $\sigma = 0.00323$, $b = 0$, $p_0 = 219.60$, $p_1 = 230.00$, $\bar{S} = 0.0774$ and $\mu r = 0.32$. The x-axis gives the surplus consumption ratio on the support of the distribution $S = [\bar{S}e^{-0.32}, \bar{S}e^{0.32}]$, The y-axis records the steady state probability distribution for the surplus consumption ratio.

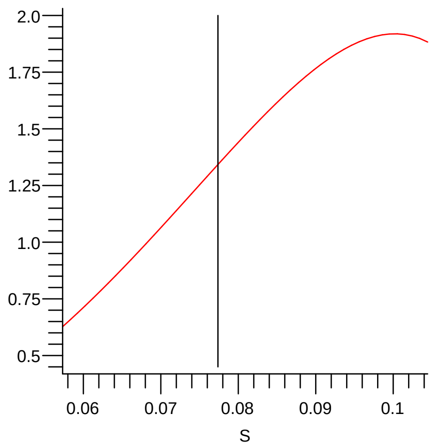

Figure 2

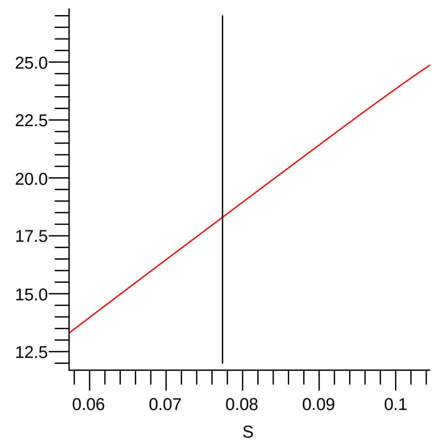

Figure 3

Figure 3 displays the price-dividend function in the Campbell and Cochrane model. The parameter values are $r^b = 0.00078$, $\bar{x} = 0.00157$, $\phi = 0.9940$, $\gamma = 3.457$, $\sigma = 0.00323$, $b = 0$, $p_0 = 219.60$, $p_1 = 230.00$, $\bar{S} = 0.0774$ and $\mu r = 0.32$. The x-axis gives the surplus consumption ratio on the support of the distribution $S = [\bar{S}e^{-0.32}, \bar{S}e^{0.32}]$. The y-axis records the price-dividend ratio.

Figure 4 displays the error analysis for the Campbell and Cochrane model. The paramet values are $r^b = 0.00078$, $\bar{x} = 0.00157$, $\phi = 0.9940$, $\gamma = 3.457$, $\sigma = 0.00323$, $b = 0$, $p_0 = 219.60$, $p_1 = 230.00$, $\bar{S} = 0.0774$ and $\mu r = 0.32$. The x-axis gives the surplus consumption ratio on the support of the distribution $S = [\bar{S}e^{-0.32}, \bar{S}e^{0.32}]$. The y-axis for the $\diamond$ line compares the 975^{th} order Taylor polynomial for the price-dividend ratio with the first order Taylor polynomial. In addition, the solid line compares the 975^{th} order Taylor polynomial for the price-dividend ratio to it's fourth order Taylor polynomial.

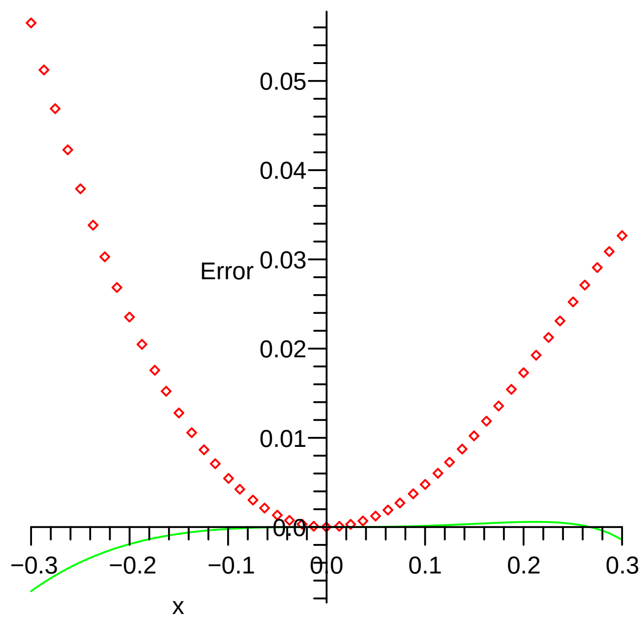

Figure 4

Figure 5 portrays the equity premium and standard deviation of equity in the continuous time model of Campbell and Cochrane. The parameter values are $r^b = 0.00078$, $\bar{x} = 0.00157$, $\phi = 0.9940$, $\gamma = 3.457$, $\sigma = 0.00323$, $b = 0$, $p_0 = 219.60$, $p_1 = 230.00$, $\bar{S} = 0.0774$ and $\mu r = 0.32$. The x-axis gives the surplus consumption ratio on the support of the distribution $S = [\bar{S}e^{-0.32}, \bar{S}e^{0.32}]$. The y-axis records the equity premium and standard deviation. The equity premium line is the bottom line, while the top line represents the standard deviation.

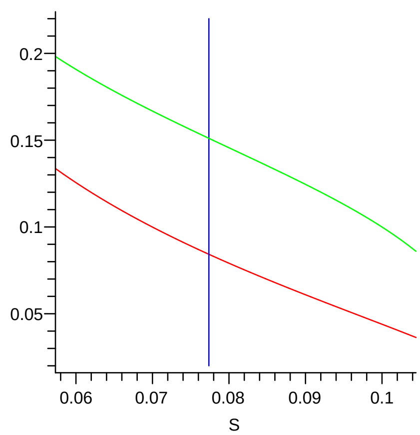

Figure 5

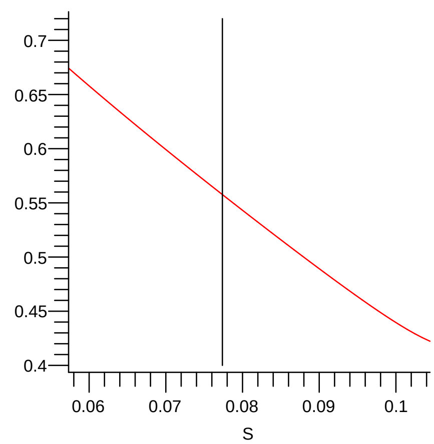

Figure 6

Figure 6 shows the Sharpe ratio in the model of Campbell and Cochrane. The parameter values are $r^b = 0.00078$, $\bar{x} = 0.00157$, $\phi = 0.9940$, $\gamma = 3.457$, $\sigma = 0.00323$, $b = 0$, $p_0 = 219.60$, $p_1 = 230.00$, $\bar{S} = 0.0774$ and $\mu r = 0.32$. The x-axis gives the surplus consumption ratio on the support of the distribution $S = [\bar{S}e^{-0.32}, \bar{S}e^{0.32}]$. The y-axis records the Sharpe ratio.

7 Appendix

Proof of Theorem 2.1. We begin by recalling our initial value problem

$$y''(x) + a(x)y'(x) + b(x)P(x) = g(x), \; y(0) = y_0, \; y'(0) = y_1. \tag{7.1}$$

Since $a(x)$, $b(x)$ and $g(x)$ are analytic about $x = 0$ with radius of convergence r_0 we have

$$a(x) = \sum_{k=0}^{\infty} a_k x^k, \quad b(x) = \sum_{k=0}^{\infty} b_k x^k, \qquad g(x) = \sum_{k=0}^{\infty} d_k x^k, \tag{7.2}$$

and for any $0 < r < r_0$, there exist $M_a, \; M_b, \; M_g > 0$ such that

$$|a_k| r^k \le M_a, \quad |b_k| r^k \le M_b, \quad |d_k| r^k \le M_g, \quad k = 0, 1, 2, \cdots . \tag{7.3}$$

Now let us assume that the solution $y(x)$ can be written (at least formally) as a power series, that is

$$y(x) = \sum_{k=0}^{\infty} c_k x^k, \tag{7.4}$$

where $c_0 = y_0$, $c_1 = y_1$ and c_k, $k = 2, 3, \cdots$ are to be determined so that $y(x)$ is a solution. We have

$$y' = \sum_{k=1}^{\infty} k c_k x^{k-1} = \sum_{k=0}^{\infty} (k+1) c_{k+1} x^k, \tag{7.5}$$

and

$$y'' = \sum_{k=2}^{\infty} k(k-1) c_k x^{k-2} = \sum_{k=0}^{\infty} (k+2)(k+1) c_{k+2} x^k. \tag{7.6}$$

For y to be a solution we must have

$$\sum_{k=0}^{\infty} (k+2)(k+1) c_{k+2} x^k + \Big(\sum_{k=0}^{\infty} a_k x^k\Big)\Big(\sum_{k=0}^{\infty} (k+1) c_{k+1} x^k\Big) + \Big(\sum_{k=0}^{\infty} b_k x^k\Big)\Big(\sum_{k=0}^{\infty} c_k x^k\Big) = \sum_{k=0}^{\infty} d_k x^k,$$

which, after multiplying the series, gives

$$\sum_{k=0}^{\infty} \Big[(k+2)(k+1) c_{k+2} + \sum_{j=0}^{k} a_{k-j} (j+1) c_{j+1} + \sum_{j=0}^{k} b_{k-j} c_j\Big] x^k = \sum_{k=0}^{\infty} d_k x^k. \tag{7.7}$$

From the last equation we obtain the recurrence relation

$$(k+2)(k+1) c_{k+2} = d_k - \sum_{j=0}^{k} \Big[a_{k-j} (j+1) c_{j+1} + b_{k-j} c_j\Big] \tag{7.8}$$

for computing the coefficients $c_2, c_3, \cdots$.

Taking absolute values in (7.8) and using the Cauchy estimates (7.3) gives

$$\begin{aligned}(k+2)(k+1)|c_{k+2}| &\le |d_k| + \sum_{j=0}^{k}\Big[|a_{k-j}|(j+1)|c_{j+1}| + |b_{k-j}||c_j|\Big]\\ &\le \frac{M_g}{r^k} + \sum_{j=0}^{k}\Big[\frac{M_a}{r^{k-j}}(j+1)|c_{j+1}| + \frac{M_b}{r^{k-j}}|c_j|\Big]\\ &\le \frac{M_g}{r^k} + \frac{M}{r^k}\sum_{j=0}^{k}\big[(j+1)|c_{j+1}| + |c_j|\big]r^j.\end{aligned}$$

Here $M \doteq \max\{M_a, M_b\}$. Adding the extra term $M|c_{k+1}|r$ (it will be helpful later) to the right-hand side of the last inequality gives

$$(k+2)(k+1)|c_{k+2}| \le \frac{M_g}{r^k} + \frac{M}{r^k}\sum_{j=0}^{k}\big[(j+1)|c_{j+1}| + |c_j|\big]r^j + M|c_{k+1}|r. \tag{7.9}$$

Letting $C_0 \doteq |c_0| = |y_0|$, $C_1 \doteq |c_1| = |y_1|$ and for $k \ge 2$ defining C_k by the recurrence relation

$$(k+2)(k+1)C_{k+2} = \frac{M_g}{r^k} + \frac{M}{r^k}\sum_{j=0}^{k}[(j+1)C_{j+1} + C_j]r^j + MC_{k+1}r\,, \tag{7.10}$$

we see that

$$|c_k| \le C_k,\ \ k = 0, 1, 2, \cdots. \tag{7.11}$$

Therefore, the series $\sum_{k=0}^{\infty} c_k x^k$ converges if $\sum_{k=0}^{\infty} C_k x^k$ does.

Next we shall show that the series $\sum_{k=0}^{\infty} C_k x^k$ converges for $|x| < r$. For this, by the ratio test, it suffices to show $\limsup_{k\to\infty} C_{k+1}/C_k \le 1/r$. In recurrence relation (7.10) replacing k with $k-1$ gives

$$(k+1)kC_{k+1} = \frac{M_g}{r^{k-1}} + \frac{M}{r^{k-1}}\sum_{j=0}^{k-1}\big[(j+1)C_{j+1} + C_j\big]r^j + MC_k r,\ \ k \ge 1, \tag{7.12}$$

and replacing k with $k-2$ gives

$$k(k-1)C_k = \frac{M_g}{r^{k-2}} + \frac{M}{r^{k-2}}\sum_{j=0}^{k-2}\big[(j+1)C_{j+1} + C_j\big]r^j + MC_{k-1}r,\ \ k \ge 2. \tag{7.13}$$

Multiplying (7.12) by r and using (7.13) gives

$$\begin{aligned}
r(k+1)kC_{k+1} &\le \frac{M_g}{r^{k-2}} + \frac{M}{r^{k-2}}\Big\{\sum_{j=0}^{k-2}\big[(j+1)C_{j+1}+C_j\big]r^j + \big[kC_k + C_{k-1}\big]r^{k-1}\Big\} + MC_kr^2 \\
&\le \frac{M_g}{r^{k-2}} + \frac{M}{r^{k-2}}\sum_{j=0}^{k-2}\big[(j+1)C_{j+1}+C_j\big]r^j + MkC_kr + MC_{k-1}r + MC_kr^2 \\
&\le \frac{M_g}{r^{k-2}} + k(k-1)C_k - \frac{M_g}{r^{k-2}} - MC_{k-1}r + MkC_kr + MC_{k-1}r + MC_kr^2.
\end{aligned}$$

From the last inequality we obtain

$$r(k+1)kC_{k+1} \le \big[k(k-1) + Mkr + Mr^2\big]C_k.$$

or

$$\frac{C_{k+1}}{C_k} \le \frac{(k-1)}{r(k+1)} + M\frac{k+r}{(k+1)k}. \tag{7.14}$$

Therefore $\limsup_{k\to\infty} C_{k+1}/C_k \le 1/r$. Thus, the function $y(x)$ defined by the power series (7.4), whose coefficients are defined by the recursion formula (7.8) has radius of convergence r_0. This justifies all operations performed above (multiplication and differentiation of series). Therefore, the solution $y(x)$ to the initial value problem (2.5) is analytic with radius r_0. $\square$

Proof of Corollary 2.2. Iterating backwards using inequality (7.14) to obtain

$$\begin{aligned}
C_k &\le C_{k-1}\left[\frac{k-2}{rk} + M\frac{k-1+r}{k(k-1)}\right] \\
&\le C_{k-2}\left[\frac{k-3}{r(k-1)} + M\frac{k-2+r}{(k-1)(k-2)}\right]\left[\frac{k-2}{rk} + M\frac{k-1+r}{k(k-1)}\right] \\
&\le C_2\prod_{l=2}^{k-1}\left[\frac{l-1}{r(l+1)} + M\frac{l+r}{(l+1)l}\right] \\
&\le \frac{1}{2}\left[M_g + |y_1|(1+r)M + |y_0|M\right]\prod_{l=2}^{k-1}\left[\frac{l-1}{r(l+1)} + M\frac{l+r}{(l+1)l}\right].
\end{aligned}$$

The last step uses the definition of C_2 in (7.12) when $k=1$.

Using this, the Taylor series remainder is estimated as follows

$$\begin{aligned}\left|y(x)-\sum_{k=0}^{n}c_kx^k\right| &= \sum_{k=n+1}^{\infty}|c_k||x|^k \leq \sum_{k=n+1}^{\infty}C_k|x|^k\\ &\leq \frac{1}{2}\left[M_g+|y_1|(1+r)M+|y_0|M\right]\sum_{k=n+1}^{\infty}\prod_{l=2}^{k-1}\left[\frac{l-1}{r(l+1)}+M\frac{l+r}{(l+1)l}\right]|x|^k\\ &\leq \frac{1}{2}\left[M_g+|y_1|(1+r)M+|y_0|M\right]\sum_{k=n+1}^{\infty}\prod_{l=2}^{k-1}\left[\frac{l-1}{r(l+1)}+M\frac{l+r}{(l+1)l}\right]|\mu r|^k.\end{aligned}$$

Consequently, we have a uniform bound for the Taylor series remainder for $|x| \leq |\mu r|$, where $0 \leq \mu \leq 1$. $\square$

Derivation of (3.3), the Stochastic Distribution of z. Arnold (1974, p. 129) equation (8.2.1) is given by

$$dX_t = \Big(A(t)X_t + a(t)\Big)dt + B(t)dW_t.$$

where dW_t is a Brownian motion. In our case $A(t) = (\psi - 1)$ is a constant and the dimension of X is one, so Corollary (8.2.5) of Arnold (1974, p. 130) applies. As a result the solution to (3.2) is

$$z(t) = \exp\left[(\psi-1)\int_0^t ds\right]\left[z(0) + \int_0^t \exp\left[-(\psi-1)\int_0^s du\right]\{-(\psi-1)\bar{z}ds + \sigma d\omega\}\right],$$

where we use the definitions $a(t) = -(\psi-1)\bar{z}$. This equation may be expressed as

$$\begin{aligned}&\exp\left[(\psi-1)t\right]\left[z(0) + \int_0^t \exp\left[-(\psi-1)s\right]\{-(\psi-1)\bar{z}ds + \sigma d\omega\}\right]\\ &= \exp\left[(\psi-1)t\right]\left[z(0) - (\psi-1)\bar{z}\int_0^t \exp\left[-(\psi-1)s\right]ds + \sigma\int_0^t \exp\left[-(\psi-1)s\right]d\omega\right].\end{aligned}$$

Now we have

$$\int_0^t \exp\left[-(\psi-1)s\right]ds = -\frac{1}{(\psi-1)}\exp\left[-(\psi-1)s\right]|_0^t = -\frac{1}{(\psi-1)}\left[\exp\left[-(\psi-1)t\right]-1\right].$$

As a result we have

$$z(t) = \exp\left[(\psi-1)t\right](z(0)-\bar{z}) + \bar{z} + \sigma\int_0^t \exp\left[(\psi-1)(t-s)\right]d\omega.$$

¿From Shreve (2004 Theorem (4.4.9), p. 149) we have that

$$\sigma\int_0^t \exp\left[(\psi-1)(t-s)\right]d\omega$$

is a random variable with normally (Gaussian) distributed random variable with expected value 0 and variance

$$\begin{aligned}
\sigma^2 \textstyle\int_0^t \{\exp[(\psi-1)(t-s)]\}^2\, ds \quad &= \sigma^2 \textstyle\int_0^t \exp[2(\psi-1)(t-s)]\, ds \\
&= \sigma^2 \exp[2(\psi-1)t] \textstyle\int_0^t \exp[-2(\psi-1)s]\, ds \\
&= -\tfrac{\sigma^2}{2(\psi-1)} \exp[2(\psi-1)t] \exp[-2(\psi-1)s]\Big|_{s=0}^{s=t} \\
&= -\tfrac{\sigma^2}{2(\psi-1)} \exp[2(\psi-1)t]\, \{\exp[-2(\psi-1)t]-1\} \\
&= \tfrac{\sigma^2}{2(\psi-1)}\, \{\exp[2(\psi-1)t]-1\}\ .
\end{aligned}$$

Thus, the variance is

$$\frac{\sigma^2}{2(1-\psi)}\left[1-\exp\left[2(\psi-1)t\right]\right].$$

By letting $t \to \infty$ we find the steady state distribution for z. $\square$

Recursive formula (4.13) for Campbell and Cochrane's model. By the Cauchy-Kovalevsky Theorem 2.1 we can write

$$P(s) = \sum_{n=0}^{\infty} p_n s^n \qquad \text{near } s = 0.$$

Also, we write $c_i(s) = \sum_{n=0}^{\infty} c_i^{(n)} s^n$ near $s = 0$. We will derive the recurrence formula for p_n with $n \geqslant 2$. For this, we calculate

$$P'(s) = \sum_{n=0}^{\infty} (n+1) p_{n+1} s^n \qquad \text{and} \qquad P''(s) = \sum_{n=0}^{\infty} (n+1)(n+2) p_{n+2} s^n\,.$$

Then, using the product formula for convergent series we obtain

$$\begin{aligned}
c_2(s)P''(s) &= \left[\sum_{n=0}^{\infty} c_2^{(n)} s^n\right]\left[\sum_{n=0}^{\infty}(n+1)(n+2)p_{n+2}s^n\right] = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}(k+1)(k+2)c_2^{(n-k)}p_{k+2}\right]s^n, \\
c_1(s)P'(s) &= \left[\sum_{n=0}^{\infty} c_1^{(n)} s^n\right]\left[\sum_{n=0}^{\infty}(n+1)p_{n+1}s^n\right] = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}(k+1)c_1^{(n-k)}p_{k+1}\right]s^n, \\
c_0(s)P(s) &= \left[\sum_{n=0}^{\infty} c_0^{(n)} s^n\right]\left[\sum_{n=0}^{\infty}p_n s^n\right] = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}c_0^{(n-k)}p_k\right]s^n.
\end{aligned}$$

Substituting the formulas for $c_2(s)P''(s)$, $c_1(s)P'(s)$, and $c_0(s)P(s)$ into the differential equation gives

$$\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}(k+1)(k+2)c_2^{(n-k)}p_{k+2}\right]s^n = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}(k+1)c_1^{(n-k)}p_{k+1} + \sum_{k=0}^{n}c_0^{(n-k)}p_k\right]s^n - 1.$$

Matching the coefficients of same powers we obtain

$$2c_2^{(0)}p_2 = c_1^{(0)}p_1 + c_0^{(0)}p_0 - 1,$$
$$\sum_{k=0}^{n}(k+1)(k+2)c_2^{(n-k)}p_{k+2} = \sum_{k=0}^{n}(k+1)c_1^{(n-k)}p_{k+1} + \sum_{k=0}^{n}c_0^{(n-k)}p_k \quad \text{for } n \geqslant 1.$$

Finally, solving the second equation for p_{n+2} $(n \geqslant 1)$ gives

$$\begin{aligned}(n+1)(n+2)c_2^{(0)}p_{n+2} &= \sum_{k=1}^{n+1} kc_1^{(n-k+1)}p_k + \sum_{k=0}^{n} c_0^{(n-k)}p_k - \sum_{k=2}^{n+1}(k-1)kc_2^{(n-k+2)}p_k \\ &= (n+1)(c_1^{(0)} - nc_2^{(1)})p_{n+1} + \sum_{k=2}^{n}\left[c_0^{(n-k)} + kc_1^{(n-k+1)} - (k-1)kc_2^{(n-k+2)}\right]p_k \\ &+ (c_0^{(n-1)} + c_1^{(n)})p_1 + c_0^{(n)}p_0\,.\end{aligned}$$

If $c_2^{(0)} \neq 0$, then the above recurrence formula calculates all the p_n with $n \geqslant 2$. $\square$

Derivation of bounds on $a(x)$, $b(x)$, and $g(x)$. To estimate the bound on the coefficients it is necessary to find a bound on the sensitivity function $\lambda(s)$ within the complex plane following CCH (2006a). Let $z = x + yi$ be a point on the circle C_r, that is $x^2 + y^2 = r^2$. Also, we assume that $r < r_0 = \frac{1-\bar{S}^2}{2}$. We write

$$u + vi = \lambda(z) = \frac{1}{\bar{S}}\sqrt{1-2z} - 1 = \frac{1}{\bar{S}}\sqrt{1-2x-2yi} - 1 \quad \text{with } u+1 \geq 0,$$

which is equivalent to

$$(u+1)^2 - v^2 = \frac{1-2x}{\bar{S}^2} \quad \text{and} \quad (u+1)v = -\frac{y}{\bar{S}^2}\,.$$

These equations imply

$$(u+1)^4 - \frac{1-2x}{\bar{S}^2}(u+1)^2 - \frac{y^2}{\bar{S}^4} = 0 \quad \text{and} \quad v^4 + \frac{1-2x}{\bar{S}^2}v^2 - \frac{y^2}{\bar{S}^4} = 0\,.$$

If $y \neq 0$, then the quadratic formula yields

$$(u+1)^2 = \frac{1-2x+\sqrt{1-4x+4r^2}}{2\bar{S}^2} \quad \text{and} \quad v^2 = \frac{-1+2x+\sqrt{1-4x+4r^2}}{2\bar{S}^2}\,.$$

Applying

$$(1-2r)^2 = 1-4r+4r^2 \leq 1-4x+4r^2 \leq 1+4r+4r^2 = (1+2r)^2\,,$$

we get

$$\frac{1-2r}{\bar{S}^2} \le (u+1)^2 \le \frac{1+2r}{\bar{S}^2} \quad \text{and} \quad 0 \le v^2 \le \frac{2r}{\bar{S}^2}\,,$$

where the first inequality implies further that

$$0 < \lambda_1(r) \doteq \frac{\sqrt{1-2r}}{\bar{S}} - 1 \le u \le \frac{\sqrt{1+2r}}{\bar{S}} - 1 \doteq \lambda_2(r).$$

So

$$0 < \lambda_1^2(r)^2 \le u^2 + v^2 \le \lambda_2^2(r) + \frac{2r}{\bar{S}^2}\,.$$

This set of inequalities can be used to establish bounds on the coefficients for z on C_r. Since

$$c_2(z) = \frac{\sigma^2}{2}\lambda^2(z)$$

we have that

$$|c_2(z)| = \frac{\sigma^2}{2}|\lambda^2(z)| \ge \frac{\sigma^2}{2}|\lambda_1^2(r)| \doteq m_2, \quad \text{for } |z| = r.$$

Also, since

$$c_1(z) = K_1 s + (\gamma-1)\sigma^2\lambda(z) + \gamma\sigma^2\lambda^2(z)$$

we have that

$$|c_1(z)| \le K_1 r + (\gamma-1)\sigma^2\sqrt{\lambda_2^2(r) + \frac{2r}{\bar{S}^2}} + \gamma\sigma^2\left[\lambda_2^2(r) + \frac{2r}{\bar{S}^2}\right] \doteq m_1, \quad \text{for } |z| = r.$$

Finally, since

$$c_0(z) = -K_0 + \gamma K_1 z + \frac{1}{2}\sigma^2\Big(\gamma\lambda(z) + \gamma - 1\Big)^2$$

we have that

$$|c_0(z)| \le K_0 + \gamma K_1 r + \frac{\sigma^2(\gamma-1)^2}{2} + \gamma(\gamma-1)\sigma^2\sqrt{\lambda_2^2(r) + \frac{2r}{\bar{S}^2}} + \frac{\gamma^2\sigma^2}{2}\left[\lambda_2^2(r) + \frac{2r}{\bar{S}^2}\right] \doteq m_0,$$

for all $|z| = r$. Thus, the bounds on the coefficients and output are

$$|a(z)| = \left|\frac{c_1(z)}{c_2(z)}\right| \le \frac{m_1}{m_2} \doteq M_a, \quad |b(z)| = \left|\frac{c_0(z)}{c_2(z)}\right| \le \frac{m_0}{m_2} \doteq M_b, \quad \text{and} \quad |g(z)| = \left|\frac{1}{c_2(z)}\right| \le \frac{1}{m_2} \doteq M_g,$$

for z on C_r.

Then, by Cauchy's integral formula we obtain

$$|a^{(k)}(0)| \le \frac{k!}{2\pi} \oint_{C_r} \frac{|a(z)|}{r^{k+1}}\, dz = \frac{M_a k!}{2\pi} \cdot \frac{2\pi r}{r^{k+1}} = \frac{M_a k!}{r^k} \qquad \text{for } k = 0, 1, 2, \dots .$$

which corresponds to the bounds in (7.3) with $|a_k| = k!|a^{(k)}(0)|$. Following the same argument for the coefficient $b(s)$ and $g(s)$, we get

$$|b^{(k)}(0)| \le \frac{M_b k!}{r^k} \quad \text{ and } |g^{(k)}(0)| \le \frac{M_g k!}{r^k} \quad \text{ for } k = 0, 1, 2, \dots . \quad \square$$

Derivation of probability distribution (4.23). Assume that the steady state distribution $\pi(s)$ satisfies the reflection barrier condition at $s = s^*$ from Cox and Miller (1965, p. 223):

$$\left\{ \frac{1}{2} \frac{d}{ds} [a(s)\pi(s)] - b(s)\pi(s) \right\} \Big|_{s=s^*} = 0 \,.$$

Then $\frac{1}{2} \frac{d^2}{ds^2} [a(s)\pi(s)] - \frac{d}{ds} [b(s)\pi(s)] = 0$ is equivalent to

$$\frac{1}{2} \frac{d}{ds} [a(s)\pi(s)] - b(s)\pi(s) = 0$$

or to the separable equation:

$$\frac{d\pi(s)}{ds} + \frac{a'(s) - 2b(s)}{a(s)} \pi(s) = 0 \,.$$

So

$$\pi(s) = c_1 \exp\left\{ -\int^s \frac{a'(v) - 2b(v)}{a(v)}\, dv \right\} \qquad \text{for some } c_1 \in \mathbb{R} \,.$$

Recall that $a(s) = \sigma^2 \lambda(s)^2$, $b(s) = (\phi - 1)s$, and $\lambda(s) = \frac{1}{S}\sqrt{1 - 2s} - 1$.

$$\begin{aligned} \int \frac{a'(s) - 2b(s)}{a(s)}\, ds = \int \frac{2\sigma^2 \lambda(s)\lambda'(s) - 2(\phi - 1)s}{\sigma^2 \lambda(s)^2}\, ds &= 2 \int \frac{\lambda'(s)}{\lambda(s)}\, ds + \frac{2(\phi - 1)}{\sigma^2} \int \frac{s}{\lambda(s)^2}\, ds \\ &= 2 \ln \lambda(s) + \frac{2(1 - \phi)}{\sigma^2} \int \frac{s}{\lambda(s)^2}\, ds \end{aligned}$$

We will calculate $\int \frac{s}{\lambda(s)^2}\, ds$ via the change of variable: $y = \lambda(s) = \frac{1}{S}\sqrt{1 - 2s} - 1$. Then we get

$s = \frac{1}{2}[1 - (\bar{S})^2(y+1)^2]$ and $ds = -(\bar{S})^2(y+1)\,dy$.

$$\begin{aligned}
\int \frac{s}{\lambda(s)^2}\,ds &= \int \frac{1}{2y^2}\left[1 - (\bar{S})^2(y+1)^2\right]\left[-(\bar{S})^2(y+1)\right]\,dy = \frac{(\bar{S})^2}{2}\int \frac{1}{y^2}\left[(\bar{S})^2(y+1)^3 - (y+1)\right]\,dy \\
&= \frac{(\bar{S})^2}{2}\int \left[(\bar{S})^2 y + 3(\bar{S})^2 + \frac{3(\bar{S})^2 - 1}{y} + \frac{(\bar{S})^2 - 1}{y^2}\right]\,dy \\
&= \frac{(\bar{S})^2}{2}\left[\frac{(\bar{S})^2}{2}y^2 + 3(\bar{S})^2 y + (3(\bar{S})^2 - 1)\ln y + \frac{1 - (\bar{S})^2}{y}\right] + C \\
&= \frac{(\bar{S})^2}{2}\left[\frac{(\bar{S})^2}{2}\lambda(s)^2 + 3(\bar{S})^2\lambda(s) + (3(\bar{S})^2 - 1)\ln\lambda(s) + \frac{1 - (\bar{S})^2}{\lambda(s)}\right] + C
\end{aligned}$$

As a result,

$$\begin{aligned}
\int \frac{a'(s) - 2b(s)}{a(s)}\,ds = {} & \frac{2\sigma^2 + (3(\bar{S})^2 - 1)(\bar{S})^2(1-\phi)}{\sigma^2}\ln\lambda(s) \\
& + \frac{(\bar{S})^2(1-\phi)}{\sigma^2}\left[\frac{(\bar{S})^2}{2}\lambda(s)^2 + 3(\bar{S})^2\lambda(s) + \frac{1 - (\bar{S})^2}{\lambda(s)}\right] + C\,.
\end{aligned}$$

Set

$$c_2 = \int_{-\infty}^{s^*}\left\{\frac{2\sigma^2 + (3(\bar{S})^2 - 1)(\bar{S})^2(1-\phi)}{\sigma^2}\ln\lambda(s) + \frac{(\bar{S})^2(1-\phi)}{\sigma^2}\left[\frac{(\bar{S})^2}{2}\lambda(s)^2 + 3(\bar{S})^2\lambda(s) + \frac{1 - (\bar{S})^2}{\lambda(s)}\right]\right\}\,ds\,.$$

Then

$$\pi(s) = \frac{1}{c_2}\left\{\frac{2\sigma^2 + (3(\bar{S})^2 - 1)(\bar{S})^2(1-\phi)}{\sigma^2}\ln\lambda(s) + \frac{(\bar{S})^2(1-\phi)}{\sigma^2}\left[\frac{(\bar{S})^2}{2}\lambda(s)^2 + 3(\bar{S})^2\lambda(s) + \frac{1 - (\bar{S})^2}{\lambda(s)}\right]\right\}$$

for $s < s^*$. Thus (4.23) is the steady state probability function for s. $\square$

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