Estimating Macroeconomic Models: A Likelihood Approach

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- We apply particle filtering to evaluate the likelihood of the model.
- We estimate a neoclassical business cycle model with investment-specific technological change and stochastic volatility.

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- We modify the filter to be more flexible with shocks.

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1. Theoretical arguments.
2. Computational evidence.

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3. Linearization complicates the identification of parameters.

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State Space Representation of the Model

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- We want to track conditional density $p\left(S_t | y^{t-1}; \gamma\right)$.

Factorization of the Likelihood

- Why?

$$
\begin{aligned}
p\left(y^T;\gamma\right) &= \prod_{t=1}^{T} p\left(y_t|y^{t-1};\gamma\right) \\
&= \prod_{t=1}^{T} \int p\left(y_t|S_t;\gamma\right) p\left(S_t|y^{t-1};\gamma\right) dS_t
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- Knowledge of $\left\{p\left(S_t|y^{t-1};\gamma\right)\right\}_{t=1}^{T}$ allows the evaluation of the likelihood of the model.

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2. Update: Bayes' theorem

$$p\left(S_t|y^t;\gamma\right) = \frac{p\left(y_t|S_t;\gamma\right) p\left(S_t|y^{t-1};\gamma\right)}{p\left(y_t|y^{t-1};\gamma\right)}$$

where:

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- Alternatives? Unscented Kalman filter, Grid Filter,...

A Law of Large Numbers

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Suppose we have $\left\{ \left\{ s^i_{t|t-1} \right\}_{i=1}^{N} \right\}_{t=1}^{T} \sim \left\{ p\left(S_t | y^{t-1}; \gamma\right) \right\}_{t=1}^{T}$.

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Then:

$$p\left(y^T; \gamma\right) \simeq \prod_{t=1}^{T} \frac{1}{N} \sum_{i=1}^{N} p\left(y_t | s_{t|t-1}^{i}; \gamma\right)$$

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Evaluating the likelihood function $\Leftrightarrow$ Drawing from density:

$$\left\{ p\left(S_t | y^{t-1}; \gamma\right) \right\}_{t=1}^{T}$$

Introducing Particles

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- Weight of each proposed particle:

$$q^i_t = \frac{p\left(y_t|s^i_{t|t-1};\gamma\right)}{\sum_{i=1}^N p\left(y_t|s^i_{t|t-1};\gamma\right)}$$

A Proposition

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- Then, $\{\widetilde{s}^i_t\}_{i=1}^N$ is a draw from $p\left(S_t|y^t;\gamma\right)$:

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- Proof: Importance sampling and Bayes' theorem.

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1. Update: We can use a draw $\left\{ s_{t|t-1}^{i} \right\}_{i=1}^{N}$ from $p\left(S_t | y^{t-1}; \gamma\right)$ to get a draw $\left\{ s_{t|t}^{i} \right\}_{i=1}^{N}$ from $p\left(S_t | y^{t}; \gamma\right)$.

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2. Forecast: We can use a draw $\left\{s_{t|t}^{i}\right\}_{i=1}^{N}$ from $p\left(S_t|y^{t};\gamma\right)$, a draw from $p\left(W_{t+1};\gamma\right)$, and $S_{t+1}=f\left(S_t,W_{t+1};\gamma\right)$ to get a draw $\left\{s_{t+1|t}^{i}\right\}_{i=1}^{N}$.

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Step 0, Initialization: Sample N values $\left\{s_{1|0}^{i}\right\}_{i=1}^{N}$ from $p(S_1; \gamma)$. Go to step 2.

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Step 3, Update: Draw $\left\{s^i_{t|t}\right\}_{i=1}^N$ with replacement from $\left\{s^i_{t|t-1}\right\}_{i=1}^N$ with probabilities $\{q^i_t\}_{i=1}^N$. If $t < T$ set $t \rightsquigarrow t+1$ and go to step 1. Otherwise stop.

Particle Filtering II

Use $\left\{\left\{s_{t|t-1}^{i}\right\}_{i=1}^{N}\right\}_{t=1}^{T}$ to compute:

$$p\left(y^{T};\gamma\right) \simeq \prod_{t=1}^{T}\frac{1}{N}\sum_{i=1}^{N} p\left(y_t | s_{t|t-1}^{i};\gamma\right)$$

We can filter, forecast, and smooth

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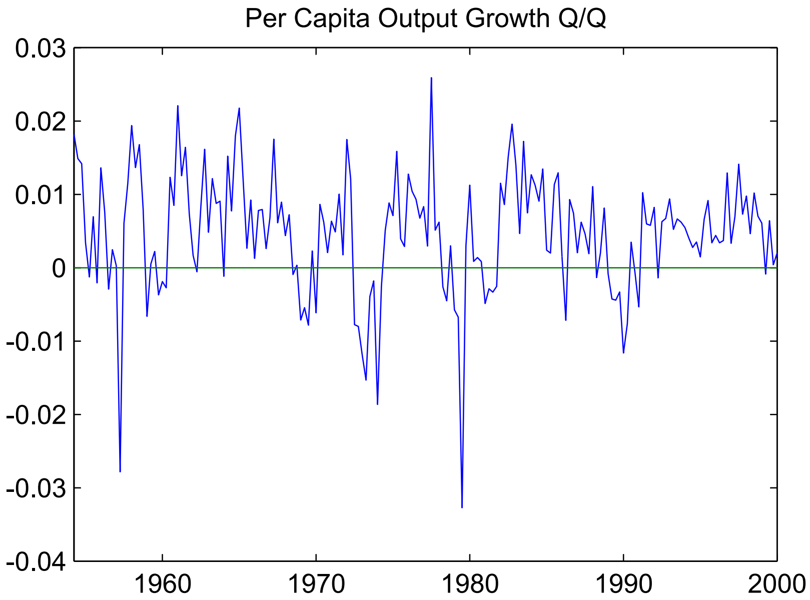
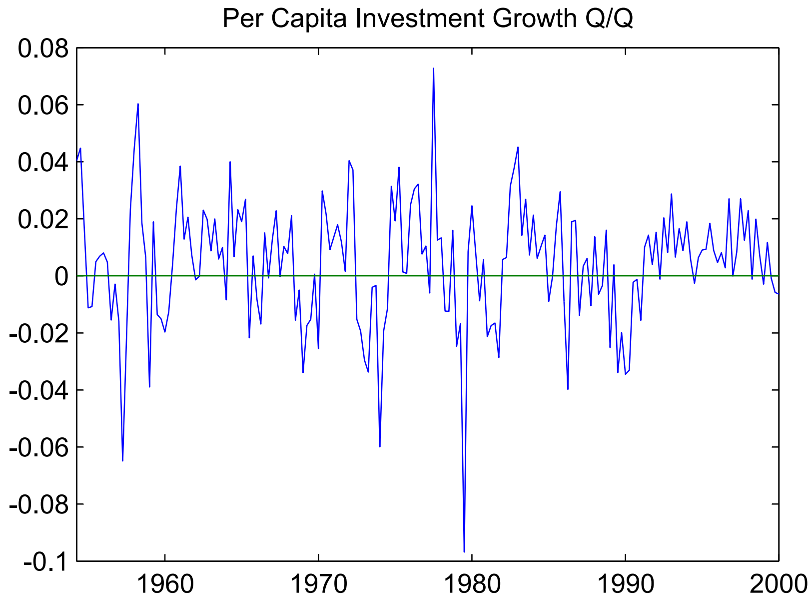
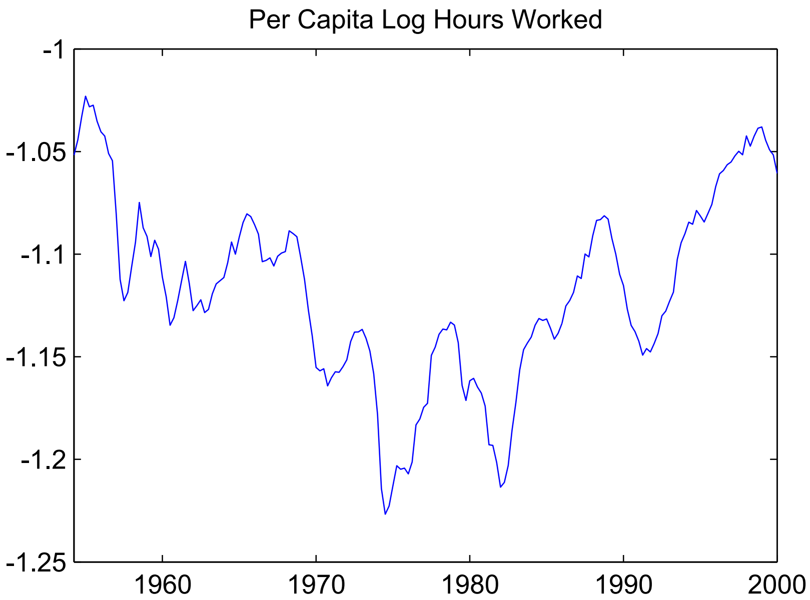
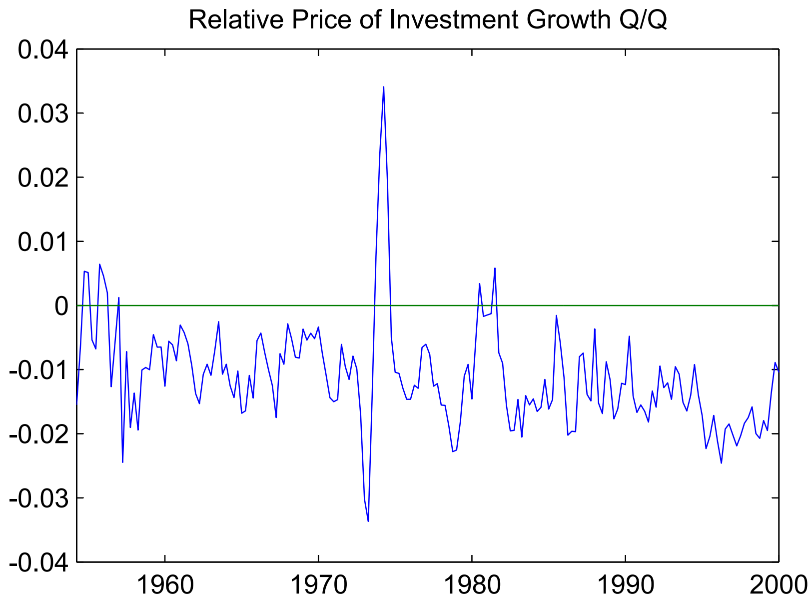

Per Capita Output Growth Q/Q
0.03
0.02
0.01
0
-0.01
-0.02
-0.03
-0.04
1960
1970
1980
1990
2000


Per Capita Investment Growth Q/Q
0.08
0.06
0.04
0.02
0
-0.02
-0.04
-0.06
-0.08
-0.1
1960
1970
1980
1990
2000


Per Capita Log Hours Worked
-1
-1.05
-1.1
-1.15
-1.2
-1.25
1960
1970
1980
1990
2000


Relative Price of Investment Growth Q/Q
0.04
0.03
0.02
0.01
0
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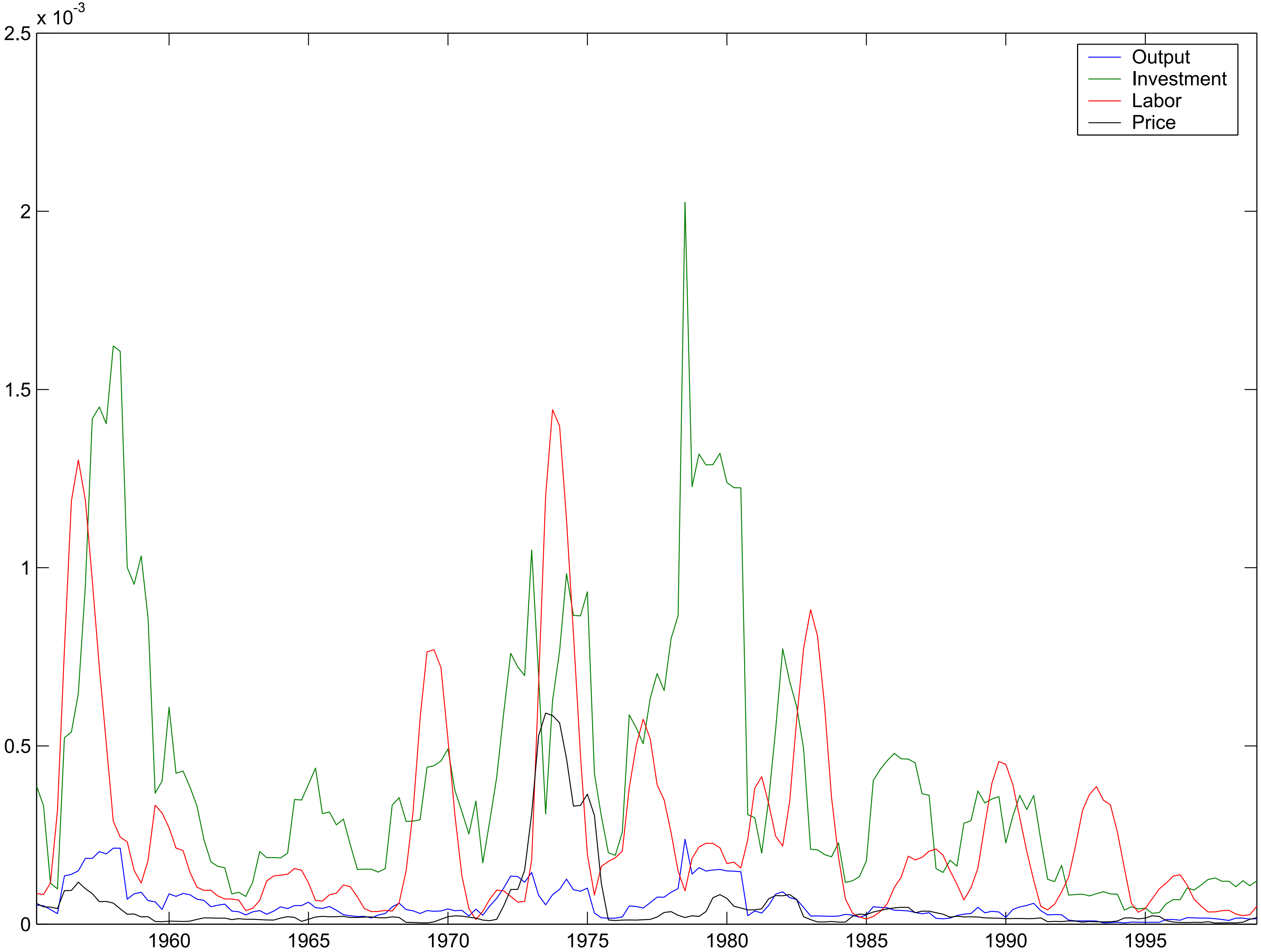
x 10^-3
2.5
2
1.5
1
0.5
0
1960
1965
1970
1975
1980
1985
1990
1995
Output
Investment
Labor
Price

Households

- Representative household with utility function:

$$E_0 \sum_{t=0}^{\infty} \beta^t \left(e^{d_t} \log C_t + \psi \log (1 - L_t) \right)$$

- Law of motion of d_t, the preference shock:

$$d_t = \rho d_{t-1} + \sigma_{dt} \varepsilon_{dt}, \; \varepsilon_{dt} \sim \mathcal{N}(0,1)$$

- We will explain later the law of motion of σ_{dt}.

Technology

- Final Good:

$$C_t + X_t = A_t K_t^{\alpha} L_t^{1-\alpha}$$

- Law of motion of capital:

$$K_{t+1} = (1-\delta) K_t + V_t X_t$$

- Shocks:

$$\begin{aligned} \log A_t &= \zeta + \log A_{t-1} + \sigma_{at}\varepsilon_{at},\ \zeta \geq 0 \text{ and } \varepsilon_{at} \sim \mathcal{N}(0,1) \\ \log V_t &= \upsilon + \log V_{t-1} + \sigma_{\upsilon t}\varepsilon_{\upsilon t},\ \upsilon \geq 0 \text{ and } \varepsilon_{\upsilon t} \sim \mathcal{N}(0,1) \end{aligned}$$

Stochastic Volatility

We follow a standard specification:

$$\log \sigma_{dt} = (1 - \lambda_d) \log \overline{\sigma}_d + \lambda_d \log \sigma_{dt-1} + \tau_d \eta_{dt} \text{ and } \eta_{dt} \sim \mathcal{N}(0, 1)$$

$$\log \sigma_{at} = (1 - \lambda_a) \log \overline{\sigma}_a + \lambda_a \log \sigma_{at-1} + \tau_a \eta_{at} \text{ and } \eta_{at} \sim \mathcal{N}(0, 1)$$

$$\log \sigma_{\upsilon t} = (1 - \lambda_\upsilon) \log \overline{\sigma}_\upsilon + \lambda_\upsilon \log \sigma_{\upsilon t-1} + \tau_\upsilon \eta_{\upsilon t} \text{ and } \eta_{\upsilon t} \sim \mathcal{N}(0, 1)$$

Analysis of the Model

- It can be shown that along a balanced growth path the following variables are stationary:

$$Y_t/Z_t,\ C_t/Z_t,\ X_t/Z_t,\ K_{t+1}/(Z_t V_t),\ (Y_t/L_t)/Z_t, \text{ and } L_t$$

  where $Z_t = A_t^{1/(1-\alpha)} V_t^{\alpha/(1-\alpha)}$.

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  where $Z_t = A_t^{1/(1-\alpha)} V_t^{\alpha/(1-\alpha)}$.
- The growth rates for the exogenous shocks are:

$$\begin{aligned} \log A_t - \log A_{t-1} &= \zeta + \sigma_{at}\varepsilon_{at} \\ \log V_t - \log V_{t-1} &= \upsilon + \sigma_{\upsilon t}\varepsilon_{\upsilon t} \end{aligned}$$

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- Therefore, Y_t, C_t, X_t, and Y_t/L_t grow at rate $(\zeta + \alpha\upsilon)/(1-\alpha)$, K_t grows at rate $(\zeta+\upsilon)/(1-\alpha)$, and L_t is stationary.

Transforming the Model

- The model is nonstationary because of the presence of two unit roots, one in each technological process.
- We need to transform the model into a stationary problem.
- We scale variables as $\widetilde{C}_t = \frac{C_t}{Z_t}$, $\widetilde{X}_t = \frac{X_t}{Z_t}$, and $\widetilde{K}_t = \frac{K_t}{Z_t V_{t-1}}$.
- Then:

$$E_0 \sum_{t=0}^{\infty} \beta^t \left(e^{d_t} \log \widetilde{C}_t + \psi \log(1 - L_t) \right)$$

$$\widetilde{C}_t + e^{\frac{\gamma + \alpha \upsilon + \sigma_{at}\varepsilon_{at} + \alpha \sigma_{\upsilon t}\varepsilon_{\upsilon t}}{1-\alpha}} \widetilde{K}_{t+1} =$$

$$e^{\gamma + \sigma_{at}\varepsilon_{at}} \widetilde{K}_t^{\alpha} L_t^{1-\alpha} + (1 - \delta)\, e^{-\upsilon - \sigma_{\upsilon t}\varepsilon_{\upsilon t}} \widetilde{K}_t$$

Equilibrium Conditions

1. Euler equation:

$$\frac{e^{d_t} e^{\frac{\gamma+\alpha\upsilon+\varepsilon_{at}+\alpha\sigma_{\upsilon t}\varepsilon_{\upsilon t}}{1-\alpha}}}{\widetilde{C}_t} =$$

$$\beta E_t \frac{e^{d_{t+1}}}{\widetilde{C}_{t+1}} \left(\alpha e^{\gamma+\sigma_{at+1}\varepsilon_{at+1}} \widetilde{K}_{t+1}^{\alpha} L_{t+1}^{1-\alpha} + (1-\delta)\, e^{-\upsilon-\sigma_{\upsilon t+1}\varepsilon_{\upsilon t+1}} \right)$$

2. A labor supply condition:

$$\psi \frac{e^{d_t} \widetilde{C}_t}{1-L_t} = (1-\alpha)\, e^{\gamma+\sigma_{at}\varepsilon_{at}} \widetilde{K}_t^{\alpha} L_t^{-\alpha}$$

3. The resource constraint:

$$\widetilde{C}_t + e^{\frac{\gamma+\alpha\upsilon+\sigma_{at}\varepsilon_{at}+\alpha\sigma_{\upsilon t}\varepsilon_{\upsilon t}}{1-\alpha}} \widetilde{K}_{t+1} = e^{\gamma+\sigma_{at}\varepsilon_{at}} \widetilde{K}_t^{\alpha} L_t^{1-\alpha} + (1-\delta)\, e^{-\upsilon-\sigma_{\upsilon t}\varepsilon_{\upsilon t}} \widetilde{K}_t$$

The Steady State

- We can find the steady state of the transformed model.

- We have a cointegration relation between output and investment in nominal terms:

$$\frac{\widetilde{X}_{ss}}{\widetilde{Y}_{ss}} = \alpha \frac{\left(e^{\frac{\zeta+\alpha\upsilon}{1-\alpha}} - (1-\delta)\, e^{-\upsilon}\right)}{\frac{\exp\left(\frac{\zeta+\alpha\upsilon}{1-\alpha}\right)}{\beta} - (1-\delta)\exp\left(-\upsilon\right)}$$

- Let

$$\log \widehat{K}_{t+1} = \log \frac{\widetilde{K}_{t+1}}{\widetilde{K}_{ss}} \text{ and } \log \widehat{X}_t = \log \frac{\widetilde{X}_t}{\widetilde{X}_{ss}}$$

Solution

- We solve the model using 2nd order perturbation.
- Other non-linear solution methods are possible. Aruoba, Fernández-Villaverde, and Rubio-Ramírez (2005).
- Structure of a 2nd order approximation:

$$\begin{aligned} \log \widehat{K}_{t+1} &= \Psi_{k1} s_t + \frac{1}{2} s_t' \Psi_{k2} s_t \\ \log \widehat{X}_t &= \Psi_{x1} s_t + \frac{1}{2} s_t' \Psi_{x2} s_t \\ \log L_t &= \Psi_{l1} s_t + \frac{1}{2} s_t' \Psi_{l2} s_t \end{aligned}$$

where:

$$s_t = \left(1, \log \widehat{K}_t, d_t, \sigma_{at}\varepsilon_{at}, \sigma_{\upsilon t}\varepsilon_{\upsilon t}, \log \sigma_{dt}, \log \sigma_{at}, \log \sigma_{\upsilon t}\right)'$$

State Space Representation I: Transition Equation

$$
\begin{aligned}
f_1(\mathcal{S}_t, W_t) &= 1 \\
f_2(\mathcal{S}_t, W_t) &= \Psi_{k1} s_t + \frac{1}{2} s_t' \Psi_{k2} s_t \\
f_3(\mathcal{S}_t, W_t) &= \left((1-\lambda_a)\log \overline{\sigma}_a + \lambda_a \log \sigma_{at} + \tau_a \eta_{at+1}\right) \varepsilon_{at+1} \\
f_4(\mathcal{S}_t, W_t) &= \left((1-\lambda_\upsilon)\log \overline{\sigma}_\upsilon + \lambda_\upsilon \log \sigma_{\upsilon t} + \tau_\upsilon \eta_{\upsilon t+1}\right) \varepsilon_{\upsilon t+1} \\
f_5(\mathcal{S}_t, W_t) &= \rho d_t + e^{(1-\lambda_d)\log \overline{\sigma}_d + \lambda_d \log \sigma_{dt} + \tau_d \eta_{dt+1}} \varepsilon_{dt+1} \\
f_6(\mathcal{S}_t, W_t) &= (1-\lambda_a)\log \overline{\sigma}_a + \lambda_a \log \sigma_{at} + \tau_a \eta_{at+1} \\
f_7(\mathcal{S}_t, W_t) &= (1-\lambda_\upsilon)\log \overline{\sigma}_\upsilon + \lambda_\upsilon \log \sigma_{\upsilon t} + \tau_\upsilon \eta_{\upsilon t+1} \\
f_8(\mathcal{S}_t, W_t) &= (1-\lambda_d)\log \overline{\sigma}_d + \lambda_d \log \sigma_{dt} + \tau_d \eta_{dt+1} \\
f_{9-16}(\mathcal{S}_t, W_t) &= s_t
\end{aligned}
$$

where $\mathcal{S}_t = (s_t, s_{t-1})$.

State Space Representation II: Measurement Equation

$$\begin{pmatrix} \Delta \log P_t \\ \Delta \log Y_t \\ \Delta \log X_t \\ \log L_t \end{pmatrix} = \begin{pmatrix} -\upsilon \\ \frac{\gamma+\alpha\upsilon}{1-\alpha} \\ \frac{\gamma+\alpha\upsilon}{1-\alpha} \\ \log L_{ss} + \Psi_{l3} \end{pmatrix} +$$

$$\begin{pmatrix} -\varepsilon_{\upsilon t} \\ \Psi_{y1}\left(s_t - s_{t-1}\right) + \frac{1}{2}\left(s_t'\Psi_{y2}s_t - s_{t-1}'\Psi_{y2}s_{t-1}\right) + \frac{\sigma_{at-1}\varepsilon_{at-1}+\alpha\sigma_{\upsilon t-1}\varepsilon_{\upsilon t-1}}{1-\alpha} \\ \Psi_{x1}\left(s_t - s_{t-1}\right) + \frac{1}{2}\left(s_t'\Psi_{x2}s_t - \frac{1}{2}s_{t-1}'\Psi_{x2}s_{t-1}\right) + \frac{\sigma_{at-1}\varepsilon_{at-1}+\alpha\sigma_{\upsilon t-1}\varepsilon_{\upsilon t-1}}{1-\alpha} \\ \Psi_{l1}s_t + \frac{1}{2}s_t'\Psi_{l2}s_t \end{pmatrix}$$

$$+ \begin{pmatrix} 0 \\ \epsilon_{1t} \\ \epsilon_{2t} \\ \epsilon_{3t} \end{pmatrix}$$

Performing Likelihood-Based Inference

- Time series:

  1. Relative price of capital, output, investment, and hours.

  2. Sample: 1955:Q1 to 2000:Q4.

- Vector of parameters γ is:

$$(\rho, \beta, \psi, \alpha, \delta, \upsilon, \zeta, \tau_d, \tau_a, \tau_\upsilon, \overline{\sigma}_d, \overline{\sigma}_a, \overline{\sigma}_\upsilon, \lambda_a, \lambda_\upsilon, \lambda_d, \sigma_1^\epsilon, \sigma_2^\epsilon, \sigma_3^\epsilon)$$

- Use a Random-walk Metropolis-Hastings to explore the likelihood: Classical and Bayesian.

Table 5.1: Maximum Likelihood Estimates

Parameter	Point Estimate	Standard Error ($x10^{-3}$)
ρ	0.967	3.743
β	0.999	0.460
ψ	2.343	6.825
υ	8.960E-003	0.828
ζ	3.594E-005	2.254
τ_a	7.120E-002	1.589
τ_υ	7.772E-003	2.940
τ_d	5.653E-002	2.034
$\overline{\sigma}_a$	4.008E-004	0.692
$\overline{\sigma}_\upsilon$	8.523E-003	0.101
$\overline{\sigma}_d$	5.016E-003	2.344
λ_a	4.460E-002	6.788
λ_υ	0.998	8.248
λ_d	0.998	2.302
$\sigma_{1\epsilon}$	1.031E-005	0.424
$\sigma_{2\epsilon}$	1.024E-004	0.495
$\sigma_{3\epsilon}$	1.110E-005	0.082

Figure 6.1: Model versus Data

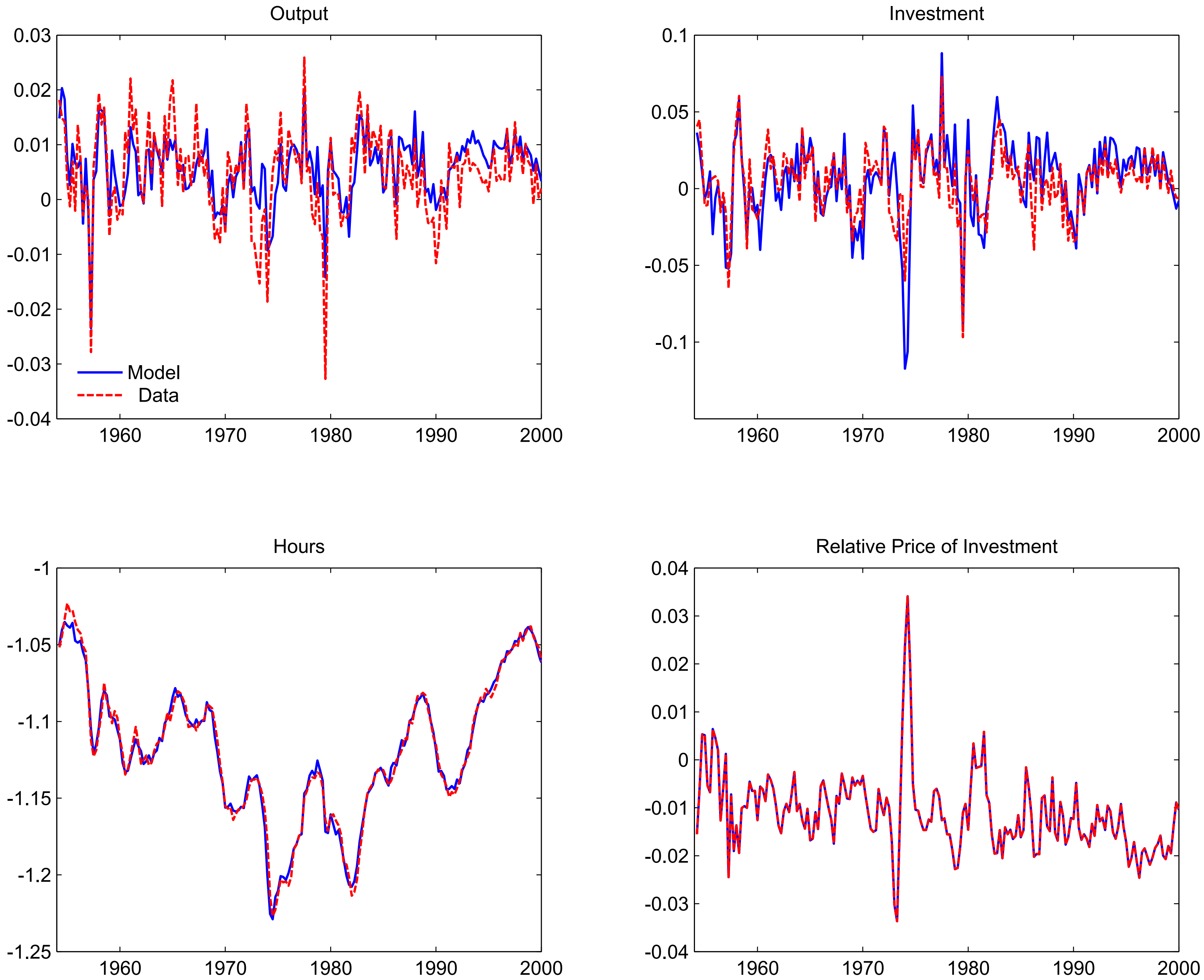


Figure 6.2: Smoothed Capital and Shocks

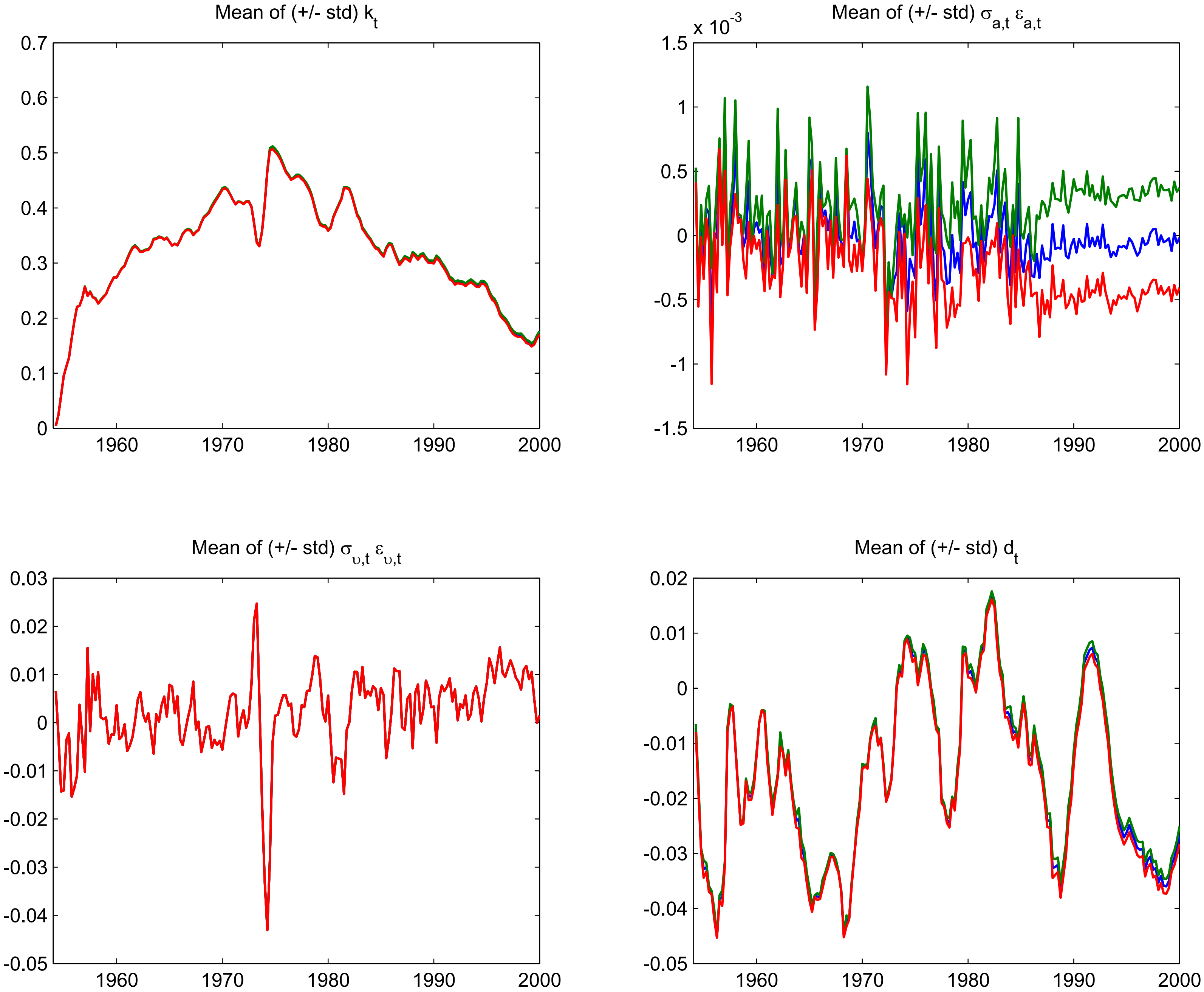


Figure 6.3: Smoothed Volatilities

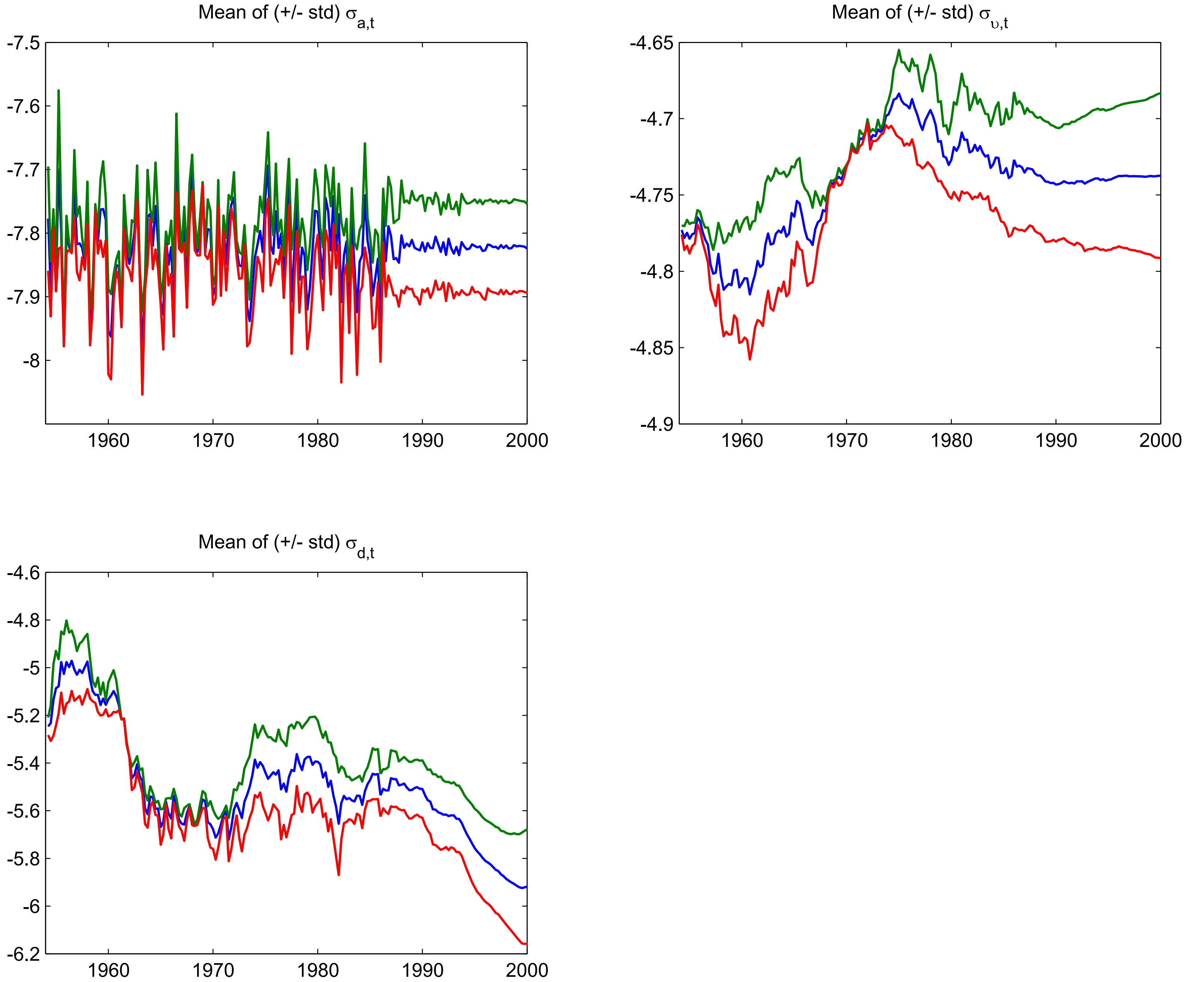


Figure 6.4: Instantaneous Standard Deviation

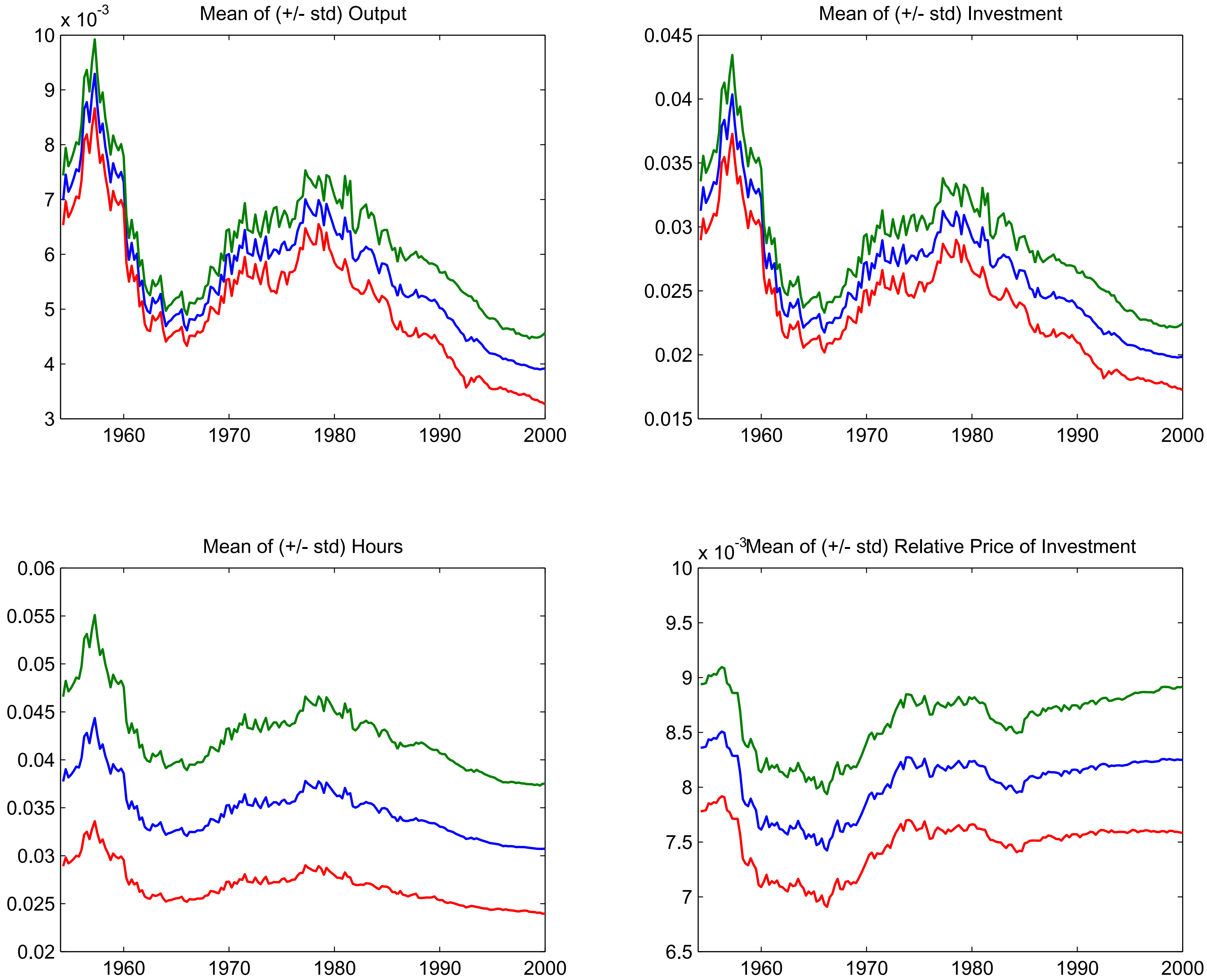


Figure 6.5: Counterfactual Exercise 1

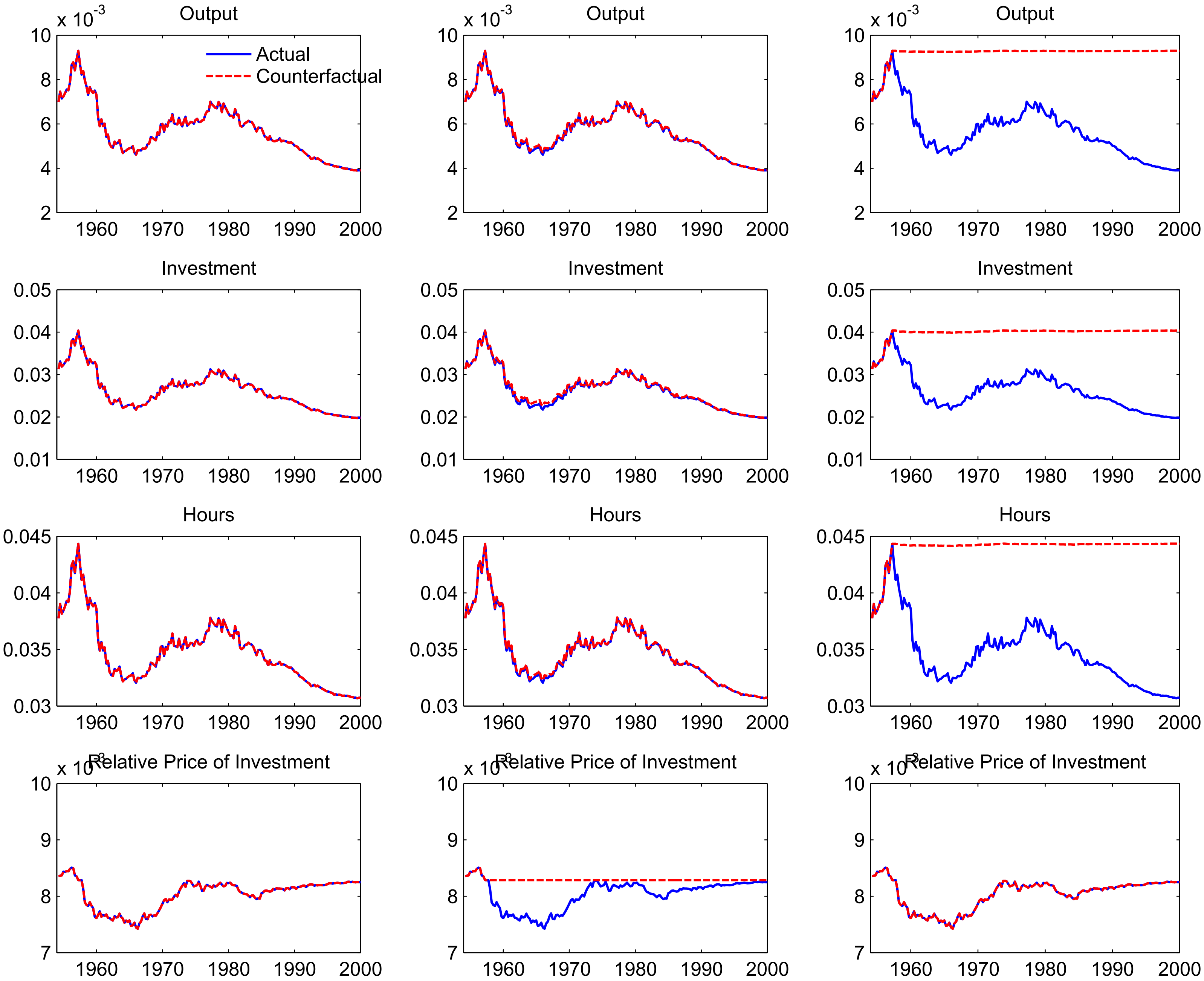


Figure 6.6: Counterfactual Exercise 2

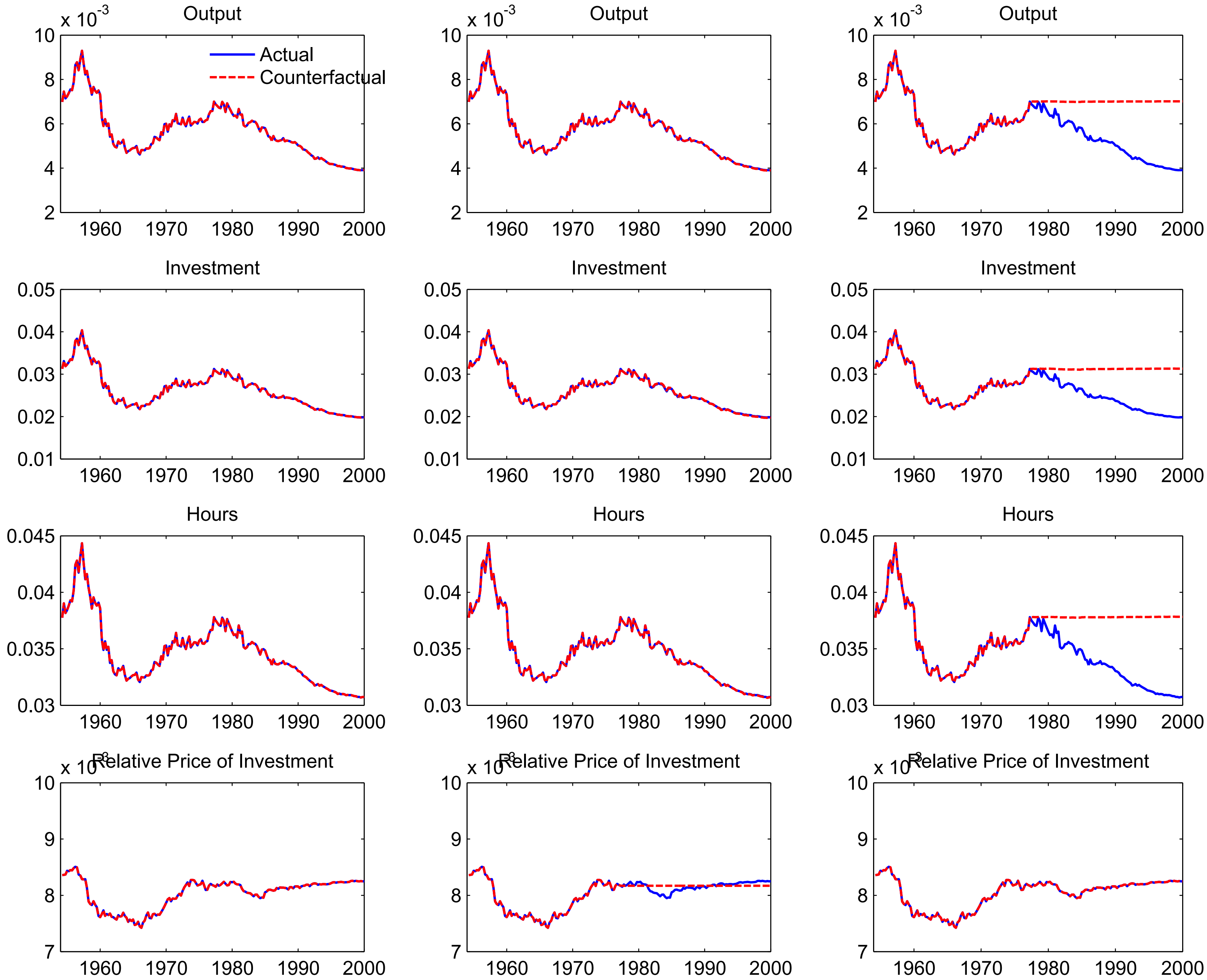
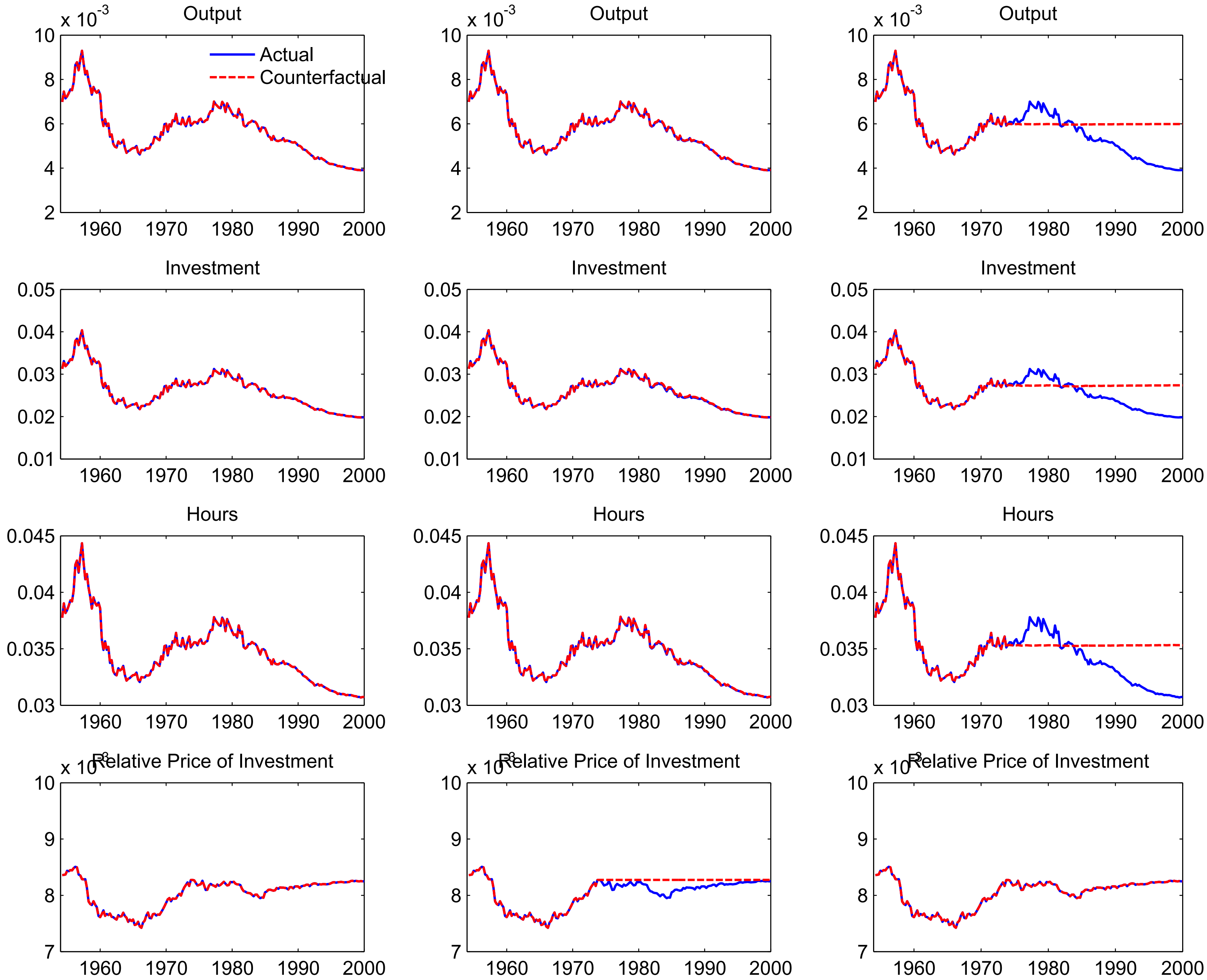


Figure 6.7: Counterfactual Exercise 3



Are Nonlinearities and Non-normalities Important?

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- We estimate four version of the model:

Table 7.1: Versions of the Model

Solution	No Stochastic Volatility	Stochastic Volatility
Linear	Version 1	Version 2
Quadratic	Version 3	Benchmark

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- Loglike benchmark: 2350.6, loglike version 2: 2230.4

Figure 7.1: Comparison of Smoothed Capital and Shocks

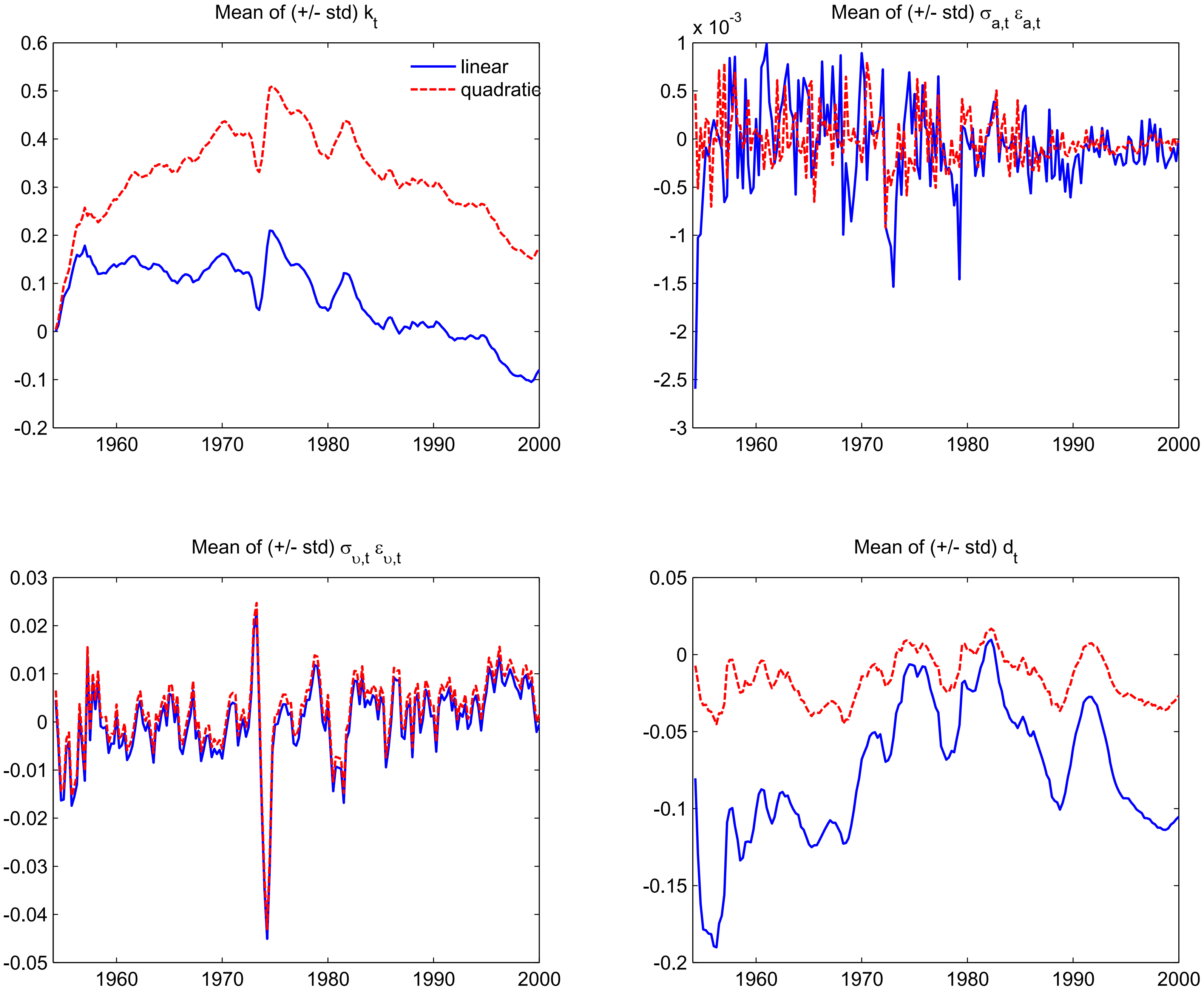
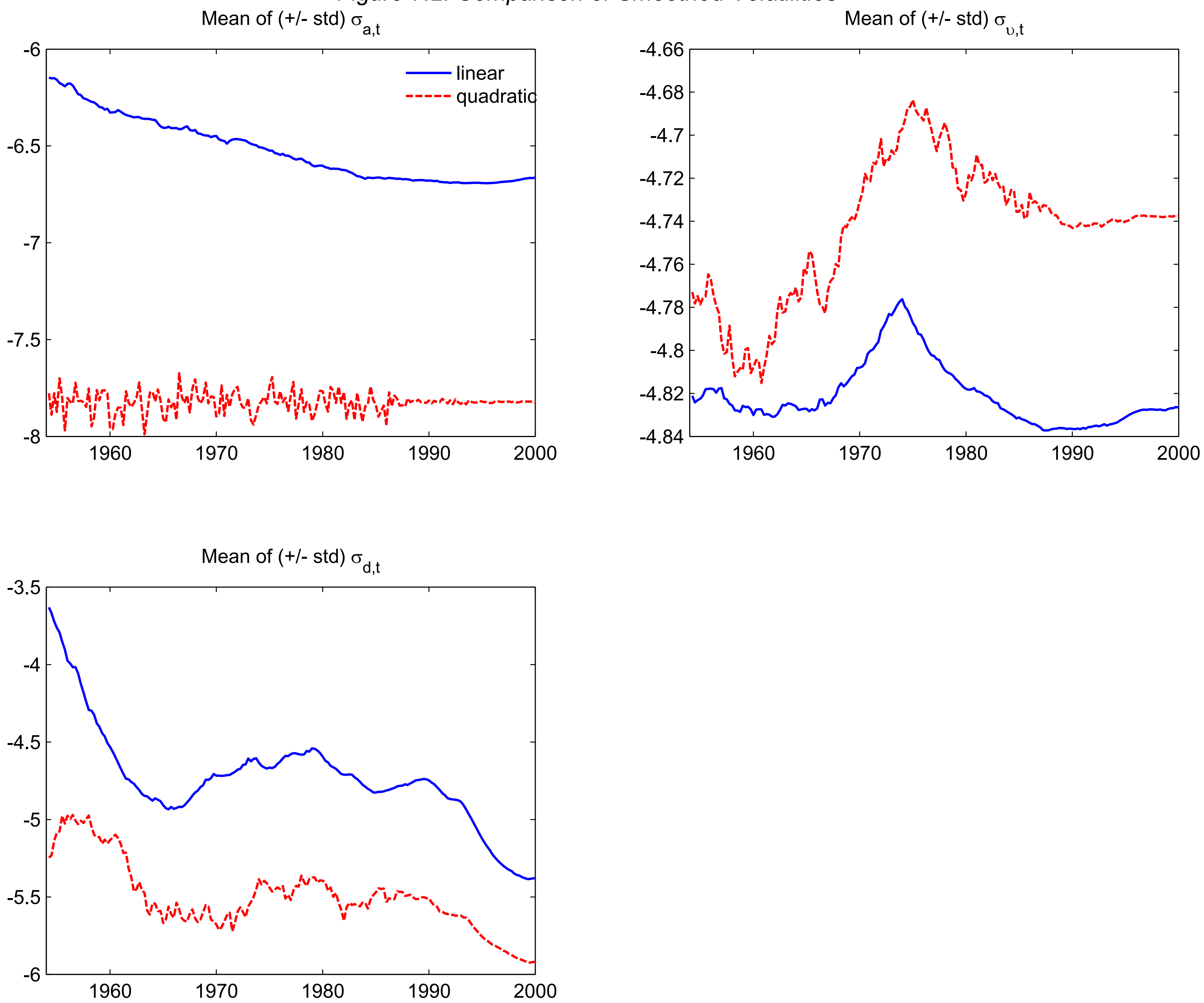


Figure 7.2: Comparison of Smoothed Volatilities



What are We Doing Now?

- We are estimating a richer DSGE model with:

  1. Nominal and real rigidities.
  2. Monetary and fiscal policy.
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- We are working on a model with micro heterogeneity.

- We are exploring the semi-nonparametric estimation of DSGE models.

Conclusions

1. Particle filtering is a general purpose and efficient method to estimate DSGE models.

2. We learned about the importance of stochastic volatility to account for U.S. Business Cycle.

3. Much exciting work to do in the next few years!