Computational Optimization for Economists

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Computational Optimization Overview

1. Introducion to Optimization [Moré]
2. Continuous Optimization in AMPL [Munson]
3. Optimization Software [Leyffer]
4. Complementarity & Games [Munson]

Part I

Introduction, Applications, and Formulations

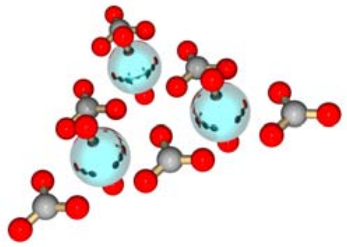
Outline: Six Topics

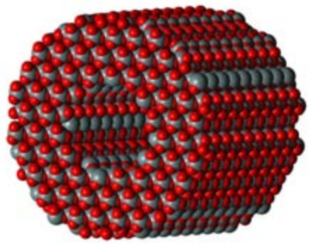
- ◇ Introduction
- ◇ Unconstrained optimization
  - Limited-memory variable metric methods
- ◇ Systems of Nonlinear Equations
  - Sparsity and Newton's method
- ◇ Automatic Differentiation
  - Computing sparse Jacobians via graph coloring
- ◇ Constrained Optimization
  - All that you need to know about KKT conditions
- ◇ Solving optimization problems
  - Modeling languages: AMPL and GAMS
  - NEOS

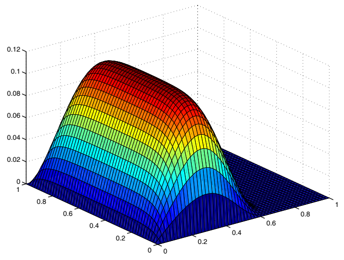
Topic 1: The Optimization Viewpoint

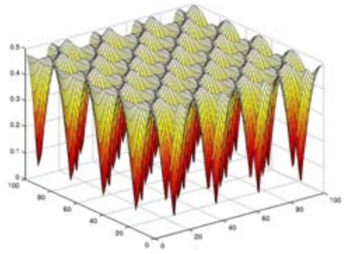
- ◇ Modeling
- ◇ Algorithms
- ◇ Software
- ◇ Automatic differentiation tools
- ◇ Application-specific languages
- ◇ High-performance architectures

View of Optimization from Applications









Classification of Constrained Optimization Problems

$$\min \{ f(x) : x_l \le x \le x_u, \; c_l \le c(x) \le c_u \}$$

- Number of variables n
- Number of constraints m
- Number of linear constraints
- Number of equality constraints n_e
- Number of degrees of freedom $n - n_e$
- Sparsity of $c'(x) = (\partial_i c_j(x))$
- Sparsity of $\nabla_x^2 \mathcal{L}(x,\lambda) = \nabla^2 f(x) + \sum_{k=1}^m \nabla^2 c_k(x) \lambda_k$

Classification of Constrained Optimization Software

- Formulation
- Interfaces: MATLAB, AMPL, GAMS
- Second-order information options:
  - Differences
  - Limited memory
  - Hessian-vector products
- Linear solvers
  - Direct solvers
  - Iterative solvers
  - Preconditioners
- Partially separable problem formulation
- Documentation
- License

Life-Cycles Saving Problem

Maximize the utility

$$\sum_{t=1}^{T} \beta^t u(c_t)$$

where S_t are the saving, c_t is consumption, w_t are wages, and

$$S_{t+1} = (1+r)S_t + w_{t+1} - c_{t+1}, \qquad 0 \le t < T$$

with $r = 0.2$ interest rate, $\beta = 0.9$, $S_0 = S_T = 0$, and

$$u(c) = -\exp(-c)$$

Assume that $w_t = 1$ for $t < R$ and $w_t = 0$ for $t \ge R$.

Question. What are the characteristics of the life-cycle problem?

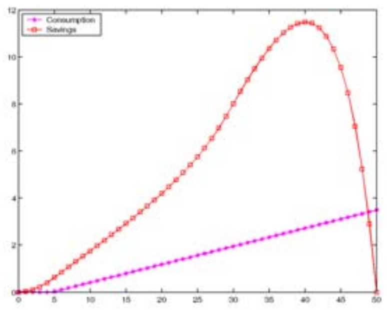
Constrained Optimization Software: IPOPT

- Formulation

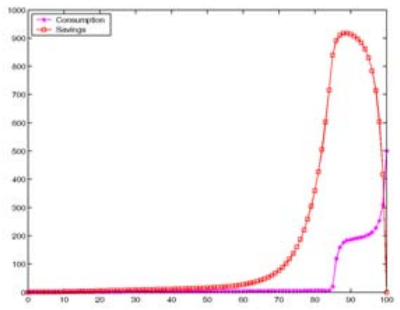
$$\min \{ f(x) : x_l \le x \le x_u,\ c(x) = 0 \}$$

- Interfaces: AMPL
- Second-order information options:
  - Differences
  - Limited memory
  - Hessian-vector products
- Direct solvers: MA27, MA57
- Partially separable problem formulation: None
- Documentation
- License

Life-Cycles Saving Problem: Results



$(R,T) = (30,50)$



$(R,T) = (60,100)$

Question. Problem formulation to results: How long?

Topic 2: Unconstrained Optimization

Augustin Louis Cauchy (August 21, 1789 – May 23, 1857)
Additional information at **Mac Tutor**
`www-history.mcs.st-andrews.ac.uk`

Unconstrained Optimization: Background

Given a continuously differentiable $f : \mathbb{R}^n \mapsto \mathbb{R}$ and

$$\min \{ f(x) : x \in \mathbb{R}^n \}$$

generate a sequence of iterates $\{x_k\}$ such that the gradient test

$$\| \nabla f(x_k) \| \le \tau$$

is eventually satisfied

Theorem. If $f : \mathbb{R}^n \mapsto \mathbb{R}$ is continuously differentiable and bounded below, then there is a sequence $\{x_k\}$ such that

$$\lim_{k \to \infty} \| \nabla f(x_k) \| = 0.$$

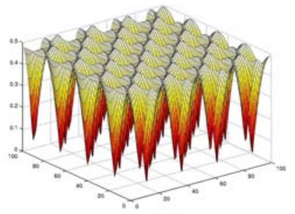
Exercise. Prove this result.

Ginzburg-Landau Model

Minimize the Gibbs free energy for a homogeneous superconductor

$$\int_{\mathcal{D}} \left\{ -|v(x)|^2 + \tfrac{1}{2}|v(x)|^4 + \| [\nabla - iA(x)]\, v(x) \|^2 + \kappa^2 \, \| (\nabla \times A)(x) \|^2 \right\} dx$$

$v : \mathbb{R}^2 \to \mathbb{C}$ (order parameter)
$A : \mathbb{R}^2 \to \mathbb{R}^2$ (vector potential)



Unconstrained problem. Non-convex function. Hessian is singular. Unique minimizer, but there is a saddle point.

Unconstrained Optimization

What can I use if the gradient $\nabla f(x)$ is not available?

- ◇ Geometry-based methods: Pattern search, Nelder-Mead, ...
- ◇ Model-based methods: Quadratic, radial-basis models, ...

What can I use if the gradient $\nabla f(x)$ is available?

- ◇ Conjugate gradient methods
- ◇ Limited-memory variable metric methods
- ◇ Variable metric methods

Computing the Gradient

Hand-coded gradients

- ◇ Generally efficient
- ◇ Error prone
- ◇ The cost is usually less than 5 function evaluations

Difference approximations

$$\partial_i f(x) \approx \frac{f((x + h e_i) - f(x)}{h_i}$$

- ◇ Choice of h_i may be problematic in the presence of noise.
- ◇ Costs n function evaluations
- ◇ Accuracy is about the $\varepsilon_f^{1/2}$ where ε_f is the noise level of f

Cheap Gradient via Automatic Differentiation

Code generated by automatic differentiation tools

- ◇ Accurate to full precision
- ◇ For the reverse mode the cost is $\Omega_T \, T\{f(x)\}$.
- ◇ In theory, $\Omega_T \le 5$.
- ◇ For the reverse mode the memory is proportional to the number of intermediate variables.

Exercise

Develop an order n code for computing the gradient of

$$f(x) = \prod_{k=1}^{n} x_k$$

Line Search Methods

A sequence of iterates $\{x_k\}$ is generated via

$$x_{k+1} = x_k + \alpha_k p_k,$$

where p_k is a descent direction at x_k, that is,

$$\nabla f(x_k)^T p_k < 0,$$

and α_k is determined by a line search along p_k.

Line searches

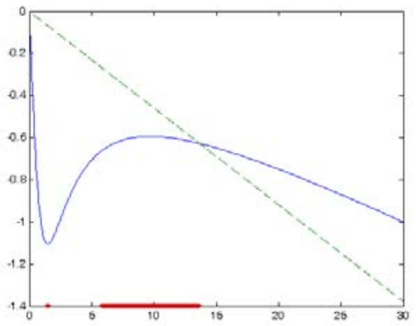
- ◇ Geometry-based: Armijo, . . .
- ◇ Model-based: Quadratics, cubic models, . . .

Powell-Wolfe Conditions on the Line Search

Given $0 \le \mu < \eta \le 1$, require that

$$f(x + \alpha p) \le f(x) + \mu \, \alpha \nabla f(x_k)^T p_k \qquad \text{sufficent decrease}$$

$$|\nabla f(x + \alpha p)^T p| \le \eta \, |\nabla f(x)^T p| \qquad \text{curvature condition}$$



Conjugate Gradient Algorithms

Given a starting vector x_0 generate iterates via

$$x_{k+1} = x_k + \alpha_k p_k$$
$$p_{k+1} = -\nabla f(x_k) + \beta_k p_k$$

where α_k is determined by a line search.

Three reasonable choices of β_k are $(g_k = \nabla f(x_k))$:

$$\beta_k^{FR} = \left(\frac{\|g_{k+1}\|}{\|g_k\|} \right)^2, \qquad \text{Fletcher-Reeves}$$

$$\beta_k^{PR} = \frac{\langle g_{k+1}, g_{k+1} - g_k \rangle}{\|g_k\|^2}, \qquad \text{Polak-Rivière}$$

$$\beta_k^{PR+} = \max\left\{\beta_k^{PR}, 0\right\}, \qquad \text{PR-plus}$$

Limited-Memory Variable-Metric Algorithms

Given a starting vector x_0 generate iterates via

$$x_{k+1} = x_k - \alpha_k H_k \nabla f(x_k)$$

where α_k is determined by a line search.

The matrix H_k is defined in terms of information gathered during the previous m iterations.

- ◇ H_k is positive definite.
- ◇ Storage of H_k requires $2mn$ locations.
- ◇ Computation of $H_k \nabla f(x_k)$ costs $(8m+1)n$ flops.

Recommendations

But what algorithm should I use?

- ◇ If the gradient $\nabla f(x)$ is not available, then a model-based method is a reasonable choice. Methods based on quadratic interpolation are currently the best choice.
- ◇ If the gradient $\nabla f(x)$ is available, then a limited-memory variable metric method is likely to produce an approximate minimizer in the least number of gradient evaluations.
- ◇ If the Hessian is also available, then a state-of-the-art implementation of Newton's method is likely to produce the best results if the problem is large and sparse.

Topic 3: Newton's Method

Library of Congress

Sir Isaac Newton (January 4, 1643 – March 331, 1727)
Additional information at **Mac Tutor**
`www-history.mcs.st-andrews.ac.uk`

Motivation

Give a continuously differentiable $f : \mathbb{R}^n \mapsto \mathbb{R}^n$, solve

$$f(x) = \begin{pmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{pmatrix} = 0$$

Linear models. The mapping defined by

$$L_k(s) = f(x_k) + f'(x_k)s$$

is a linear model of f near x_k, and thus it is sensible to choose s_k such that $L_k(s_k) = 0$ provided $x_k + s_k$ is near x_k.

Newton's Method

Given a starting point x_0, Newton's method generates iterates via

$$f'(x_k)s_k = -f(x_k), \qquad x_{k+1} = x_k + s_k.$$

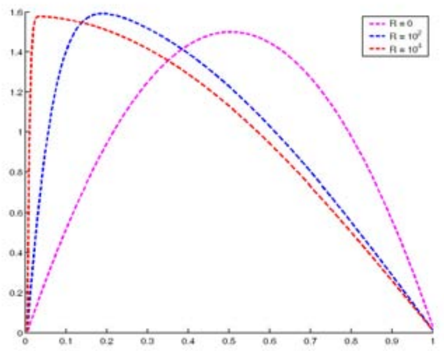
Computational Issues

- ◇ How do we solve for s_k?
- ◇ How do we handle a (nearly) singular $f'(x_k)$?
- ◇ How do we enforce convergence if x_0 is not near a solution?
- ◇ How do we compute/approximate $f'(x_k)$?
- ◇ How accurately do we solve for s_k?
- ◇ Is the algorithm scale invariant?
- ◇ Is the algorithm mesh-invariant?

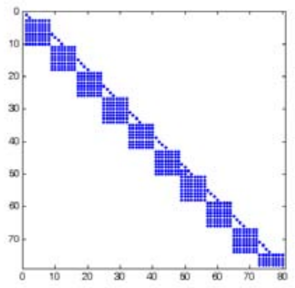
Flow in a Channel Problem

Analyze the flow of a fluid during injection into a long vertical channel, assuming that the flow is modeled by the boundary value problem below, where u is the potential function and R is the Reynolds number.

$$u'''' = R\,(u'u'' - uu''')$$
$$u(0) = 0, \quad u(1) = 1$$
$$u'(0) = u'(1) = 0$$



Sparsity



Assume that the Jacobian matrix is sparse, and let ρ_i be the number of non-zeroes in the i-th row of $f'(x)$.

- ◇ Sparse linear solvers can solve $f'(x)s = -f(x)$ in order ρ_A operations, where $\rho_A = \text{avg}\{\rho_i^2\}$.
- ◇ Graph coloring techniques (see Topic 4) can compute or approximate the Jacobian matrix with ρ_M function evaluations where $\rho_M = \max\{\rho_i\}$

Topic 4: Automatic Differentiation

Gottfried Wilhelm Leibniz (July 1, 1646 – November 14, 1716)
Additional information at **Mac Tutor**
`www-history.mcs.st-andrews.ac.uk`

Computing Gradients and Sparse Jacobians

Theorem. Given $f : \mathbb{R}^n \mapsto \mathbb{R}^m$, automatic differentiation tools compute $f'(x)v$ at a cost comparable to $f(x)$

Tasks

- Given $f : \mathbb{R}^n \mapsto \mathbb{R}^m$ with a sparse Jacobian, compute $f'(x)$ with $p \ll n$ evaluations of $f'(x)v$
- Given a partially separable $f : \mathbb{R}^n \mapsto \mathbb{R}$, compute $\nabla f(x)$ with $p \ll n$ evaluations of $\langle \nabla f(x), v \rangle$

Requirements:

$$T\{f'(x)\} \le \Omega_T \, T\{f(x)\}, \qquad M\{\nabla f(x)\} \le \Omega_M \, M\{f(x)\}$$

where $T\{\cdot\}$ is computing time and $M\{\cdot\}$ is memory.

Structurally Orthogonal Columns

Structurally orthogonal columns do not have a nonzero in the same row position.

Observation.

We can compute the columns in a group of structurally orthogonal columns with an evaluation of $f'(x)v$.

$$f'(x) = \begin{pmatrix} \times & \times & & & & \\ \times & \times & \times & & & \\ & \times & \times & \times & & \\ & & \times & \times & \times & \\ & & & \times & \times & \times \\ & & & & \times & \times \end{pmatrix}, \qquad v = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Coloring the Jacobian matrix $f'(x)$

Partitioning the columns of $f'(x)$ into p groups of structurally orthogonal columns is equivalent to a **graph coloring** problem.

For each group of structurally orthogonal columns, define $v \in \mathbb{R}^n$ with $v_i = 1$ if column i is in the group, and $v_i = 0$ otherwise. Set

$$V = (v_1, v_2, \ldots, v_p)$$

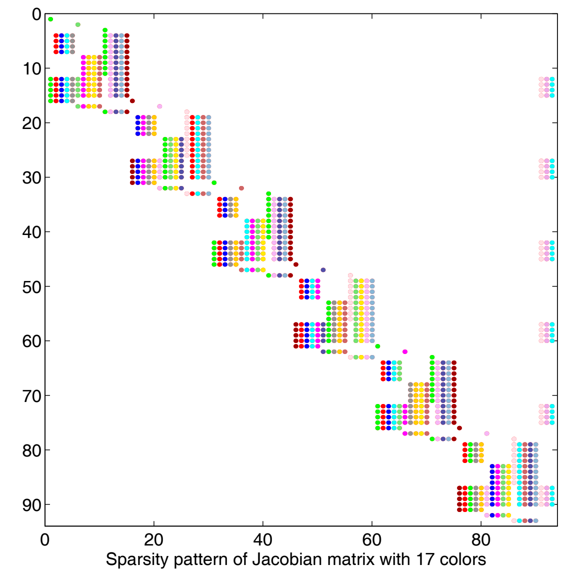
Compute $f'(x)$ from the compressed Jacobian matrix $f'(x)V$.

Observation. In practice $p \approx \rho_M$ where

$$\rho_M \equiv \max\{\rho_i\},$$

and ρ_i is the number of non-zeros in the i-th row of $f'(x)$.

Coloring the Jacobian matrix with $p = 17$ colors



Sparsity pattern of Jacobian matrix with 17 colors

Sparsity Pattern of the Jacobian Matrix

Optimization software tends to require the **closure** of the sparsity pattern

$$\bigcup \{ \mathcal{S}(f'(x)) : x \in \mathcal{D} \} .$$

in a region $\mathcal{D}$ of interest. In our case,

$$\mathcal{D} = \{ x \in \mathbb{R}^n : x_l \le x \le x_u \}$$

Given $x_0 \in \mathcal{D}$, we evaluate the sparsity pattern of ${f_E}'(\bar{x}_0)$, where $\bar{x}_0$ is a random, small perturbation of x_0, for example,

$$\bar{x}_0 = (1+\varepsilon)x_0 + \varepsilon, \qquad \varepsilon \in [10^{-6}, 10^{-4}]$$

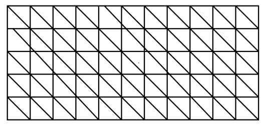
Partially Separable Functions

The mapping $f : \mathbb{R}^n \to \mathbb{R}$ is partially separable if

$$f(x) = \sum_{i=1}^{m} f_i(x),$$

and f_i only depends on $p_i \ll n$ variables.

Theorem (Griewank and Toint [1981]). If $f : \mathbb{R}^n \to \mathbb{R}$ has a sparse Hessian matrix then f is partially separable.



Optimization problems with a finite element formulation usually associate f_i with each element.

Partially Separable Functions: The Trick

If $f : \mathbb{R}^n \to \mathbb{R}$ is partially separable, the **extended** function

$$f_E(x) = \begin{pmatrix} f_1(x) \\ \vdots \\ f_m(x) \end{pmatrix}$$

has a sparse Jacobian matrix ${f_E}'(x)$. Moreover,

$$f(x) = f_E(x)^T e \qquad \Longrightarrow \qquad \nabla f(x) = {f_E}'(x)^T e$$

Observation. We can compute the dense gradient by computing the sparse Jacobian matrix ${f_E}'(x)$.

Computational Experiments

Experiments based on the MINPACK-2 collection of large-scale problems show that gradients of partially separable functions can be computed efficiently.

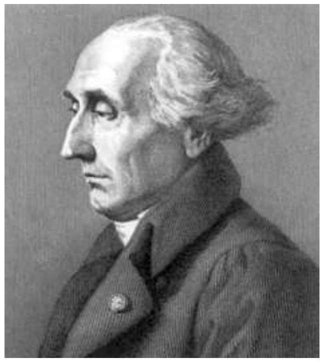
$$T\{\nabla f(x)\} = \kappa \; \rho_M \max T\{f(x)\}$$

Quartiles of κ

$2,500 \le n \le 40,000$

1.3	2.9	5.0	8.2	22.2

Topic 5: Constrained Optimization



Joseph-Louis Lagrange (January 25, 1736 – April 10, 1813)
Additional information at **Mac Tutor**
`www-history.mcs.st-andrews.ac.uk`

Geometric Viewpoint of the KKT Conditions

For any closed set Ω, consider the abstract problem

$$\min\{f(x) : x \in \Omega\}$$

The tangent cone

$$T(x^*) = \left\{ v : v = \lim_{k\to\infty} \frac{x_k - x^*}{\alpha_k},\; x_k \in \Omega,\; \alpha_k \ge 0 \right\}$$

The normal cone

$$N(x^*) = \{ w : \langle w, v \rangle \le 0,\; v \in T(x^*) \}$$

First order conditions

$$-\nabla f(x^*) \in N(x^*)$$

Computational Viewpoint of the KKT Conditions

In the case $\Omega = \{x \in \mathbb{R}^n : c(x) \ge 0\}$, define

$$C(x^*) = \left\{ w : w = \sum_{i=1}^{m} \lambda_i \left(-\nabla c_i(x^*)\right),\; \lambda_i \ge 0 \right\}$$

In general $C(x^*) \subset N(x^*)$, and under a **constraint qualification**

$$C(x^*) = N(x^*)$$

Hence, for some multipliers $\lambda_i \ge 0$,

$$\nabla f(x) = \sum_{i=1}^{m} \lambda_i \nabla c_i(x), \qquad \lambda_i \ge 0,$$

Constraint Qualifications

In the case where

$$\Omega = \{x \in \mathbb{R}^n : l \le c(x) \le u\}$$

the main two constraint qualifications are

Linear independence
The active constraint normals are positively linearly independent, that is, if

$$C_{\mathcal{A}} = \left(\nabla c_i(x) : c_i(x) \in \{l_i, u_i\}\right)$$

then $C_{\mathcal{A}}$ has full rank.

Mangasarian-Fromovitz
The active constraint normals are positively linearly independent.

Lagrange Multipliers

For the general problem with 2-sided constraints

$$\min \{ f(x) : l \le c(x) \le u \}$$

the KKT conditions for a local minimizer are

$$\nabla f(x) = \sum_{i=1}^{m} \lambda_i \nabla c_i(x), \qquad l \le c(x) \le u,$$

where the multipliers satisfy complementarity conditions

- ◇ λ_i is unrestricted if $l_i = u_i$.
- ◇ $\lambda_i = 0$ if $c_i(x) \notin \{l_i, u_i\}$
- ◇ $\lambda_i \ge 0$ if $c_i(x) = l_i$
- ◇ $\lambda_i \le 0$ if $c_i(x) = u_i$

Lagrangians

The KKT conditions for the problem with constraints $l \le c(x) \le u$ can be written in terms of the Lagrangian

$$\mathcal{L}(x, \lambda) = f(x) - \sum_{i=1}^{m} \lambda_i c_i(x).$$

Examples.

The KKT conditions for the equality-constrained $c(x) = 0$ are

$$\nabla_x \mathcal{L}(x, \lambda) = 0, \qquad c(x) = 0.$$

The KKT conditions for the inequality-constrained $c(x) \ge 0$ are

$$\nabla_x \mathcal{L}(x, \lambda) = 0, \qquad c(x) \ge 0, \quad \lambda \ge 0, \quad \lambda \perp c(x)$$

where $\lambda \perp c(x)$ means that $\lambda_i c_i(x) = 0$.

Newton's Method: Equality-Constrained Problems

The KKT conditions for the equality-constrained problem $c(x) = 0$,

$$\nabla_x \mathcal{L}(x, \lambda) = \nabla f(x) - \sum_{i=1}^{m} \lambda_i \nabla c_i(x) = 0, \qquad c(x) = 0.$$

are a system of $n + m$ nonlinear equations.

Newton's method for this system can be written as

$$x_+ = x + s_x, \qquad \lambda_+ = \lambda + s_\lambda$$

where

$$\begin{pmatrix} \nabla_x^2 \mathcal{L}(x, \lambda) & -\nabla c(x) \\ \nabla c(x)^T & 0 \end{pmatrix} \begin{pmatrix} s_x \\ s_\lambda \end{pmatrix} = - \begin{pmatrix} \nabla_x \mathcal{L}(x, \lambda) \\ c(x) \end{pmatrix}$$

Saddle Point Problems

Given a symmetric $n \times n$ matrix H and a $n \times m$ matrix C, under what conditions is

$$A = \begin{pmatrix} H & C \\ C^T & 0 \end{pmatrix}$$

nonsingular?

Lemma. If C has full rank and

$$C^T u = 0, \ u \ne 0, \qquad \Longrightarrow \qquad u^T H u > 0$$

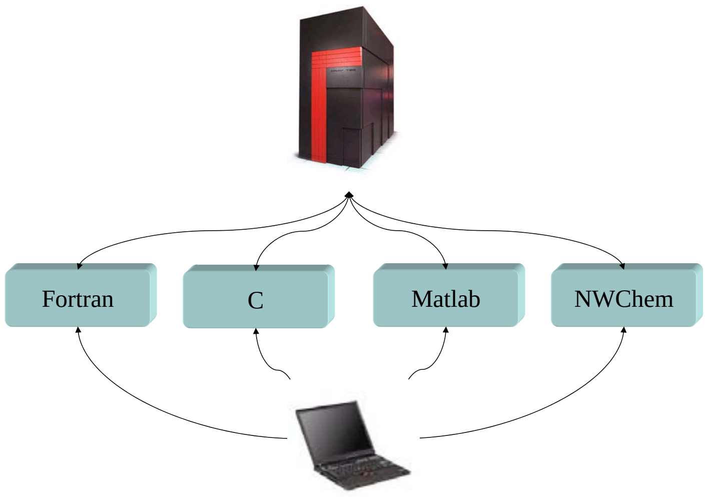
then A is nonsingular.

Topic 6: Solving Optimization Problems

Environments

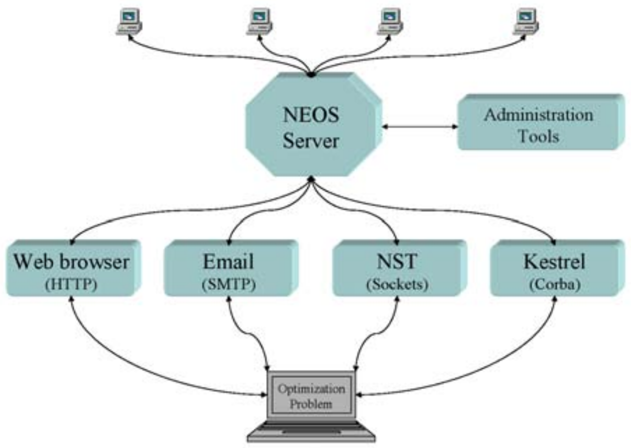
- ◇ Modeling Languages: AMPL, GAMS
- ◇ NEOS

The Classical Model



The NEOS Model

A collaborative research project that represents the efforts of the optimization community by providing access to 50+ solvers from both academic and commercial researchers.



NEOS: Under the Hood

- ◇ Modeling languages for optimization: AMPL, GAMS
- ◇ Automatic differentiation tools: ADIFOR, ADOL-C, ADIC
- ◇ Python
- ◇ Optimization solvers (50+)
  - Benchmark, GAMS/AMPL (Multi-Solvers)
  - MINLP, FortMP, GLPK, Xpress-MP, . . .
  - CONOPT, FILTER, IPOPT, KNITRO, LANCELOT, LOQO, MINOS, MOSEK, PATHNLP, PENNON, SNOPT
  - BPMPD, FortMP, MOSEK, OOQP, Xpress-MP, . . .
  - CSDP, DSDP, PENSDPP, SDPA, SeDuMi, . . .
  - BLMVM, L-BFGS-B, TRON, . . .
  - MILES, PATH
  - Concorde

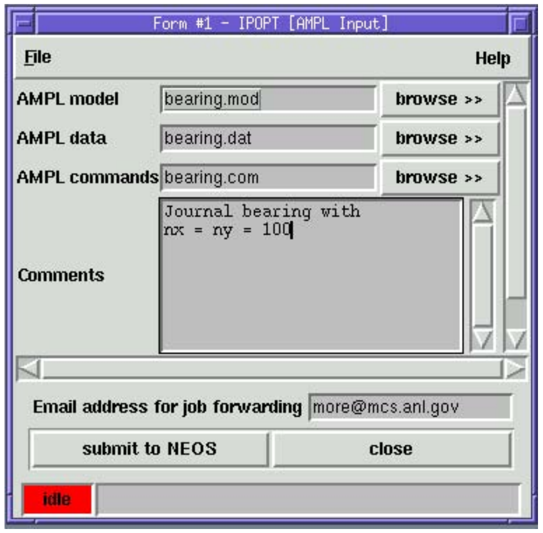
Research Issues for NEOS

- ◇ How do we add solvers?
- ◇ How are problems specified?
- ◇ How are problems submitted?
- ◇ How are problems scheduled for solution?
- ◇ How are the problems solved?
- ◇ Where are the problems solved?
  - Arizona State University
  - Lehigh University
  - Universidade do Minho, Portugal
  - Technical University Aachen, Germany
  - National Taiwan University, Taiwan
  - Northwestern University
  - Universitá di Roma *La Sapienza*, Italy
  - Wisconsin University

Solving Optimization Problems: NEOS Interfaces

Interfaces

- Kestrel
- NEOS Submit
- Web browser
- Email



Form #1 - IPOPT [AMPL Input]

File Help

AMPL model: bearing.mod browse >>
AMPL data: bearing.dat browse >>
AMPL commands: bearing.com browse >>
Comments: Journal bearing with nx = ny = 100

Email address for job forwarding: more@mcs.anl.gov

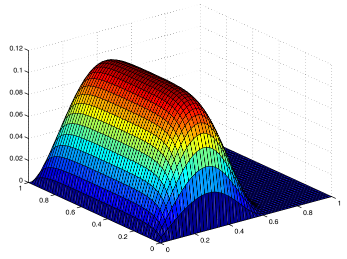
submit to NEOS close

idle

Pressure in a Journal Bearing

$$\min \left\{ \int_{\mathcal{D}} \left\{ \tfrac{1}{2} w_q(x) \|\nabla v(x)\|^2 - w_l(x) v(x) \right\} \, dx : v \geq 0 \right\}$$

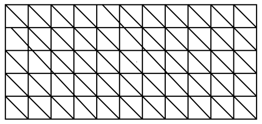
$$w_q(\xi_1, \xi_2) = (1 + \epsilon \cos \xi_1)^3$$
$$w_l(\xi_1, \xi_2) = \epsilon \sin \xi_1$$
$$\mathcal{D} = (0, 2\pi) \times (0, 2b)$$



Number of active constraints depends on the choice of ϵ in $(0,1)$.
Nearly degenerate problem. Solution $v \notin C^2$.

AMPL Model for the Journal Bearing: Parameters

Finite element triangulation



```
param nx > 0, integer; # grid points in 1st direction
param ny > 0, integer; # grid points in 2nd direction

param b;               # grid is (0,2*pi)x(0,2*b)
param e;               # eccentricity

param pi := 4*atan(1);
param hx := 2*pi/(nx+1);  # grid spacing
param hy := 2*b/(ny+1);   # grid spacing
param area := 0.5*hx*hy;  # area of triangle

param wq {i in 0..nx+1} := (1+e*cos(i*hx))^3;
```

AMPL Model for the Journal Bearing

```
var v {i in 0..nx+1, 0..ny+1} >= 0;

minimize q:
  0.5*(hx*hy/6)*sum {i in 0..nx, j in 0..ny}
    (wq[i] + 2*wq[i+1])*
    (((v[i+1,j]-v[i,j])/hx)^2 + ((v[i,j+1]-v[i,j])/hy)^2) +
  0.5*(hx*hy/6)*sum {i in 1..nx+1, j in 1..ny+1}
    (wq[i] + 2*wq[i-1])*
    (((v[i-1,j]-v[i,j])/hx)^2 + ((v[i,j-1]-v[i,j])/hy)^2) -
  hx*hy*sum {i in 0..nx+1, j in 0..ny+1} (e*sin(i*hx)*v[i,j]);

subject to c1 {i in 0..nx+1}: v[i,0] = 0;
subject to c2 {i in 0..nx+1}: v[i,ny+1] = 0;
subject to c3 {j in 0..ny+1}: v[0,j] = 0;
subject to c4 {j in 0..ny+1}: v[nx+1,j] = 0;
```

AMPL Model for the Journal Bearing: Data

```
# Set the design parameters

param b := 10;
param e := 0.1;

# Set parameter choices

let nx := 50;
let ny := 50;

# Set the starting point.

let {i in 0..nx+1,j in 0..ny+1} v[i,j]:= max(sin(i*hx),0);
```

AMPL Model for the Journal Bearing: Commands

```
option show_stats 1;

option solver "knitro";
option solver "snopt";
option solver "loqo";
option loqo_options "outlev=2 timing=1 iterlim=500";

model;
include bearing.mod;

data;
include bearing.dat;

solve;

printf {i in 0..nx+1,j in 0..ny+1}: "%21.15e\n", v[i,j] > cops.dat;
printf "%10d\n %10d\n", nx, ny > cops.dat;
```

Life-Cycles Saving Problem

Maximize the utility

$$\sum_{t=1}^{T} \beta^t u(c_t)$$

where S_t are the saving, c_t is consumption, w_t are wages, and

$$S_{t+1} = (1+r)S_t + w_{t+1} - c_{t+1}, \qquad 0 \le t < T$$

with $r = 0.2$ interest rate, $\beta = 0.9$, $S_0 = S_T = 0$, and

$$u(c) = -\exp(-c)$$

Assume that $w_t = 1$ for $t < R$ and $w_t = 0$ for $t \ge R$.

Life-Cycles Saving Problem: Model

```
param T integer;                # Number of periods
param R integer;                # Retirement
param beta;                     # Discount rate
param r;                        # Interest rate
param S0;                       # Initial savings
param ST;                       # Final savings
param w{1..T};                  # Wages

var S{0..T};                    # Savings
var c{0..T};                    # Consumption

maximize utility: sum{t in 1..T} beta^t*(-exp(-c[t]));
subject to budget {t in 0..T-1}: S[t+1] = (1+r)*S[t] + w[t+1] - c[t+1];
subject to savings {t in 0..T}: S[t] >= 0.0;
subject to consumption {t in 1..T}: c[t] >= 0.0;

subject to bc1: S[0] = S0;
subject to bc2: S[T] = ST;
subject to bc3: c[0] = 0.0;
```

Life-Cycles Saving Problem: Data

```
param T    := 100;
param R    := 60;
param beta := 0.9;
param r    := 0.2;
param S0   := 0.0;
param ST   := 0.0;

# Wages

let {i in 1..R} w[i] := 1.0;
let {i in R..T} w[i] := 0.0;

let {i in 1..R} w[i] := (i/R);
let {i in R..T} w[i] :=  (i - T)/(R - T);
```

Life-Cycles Saving Problem: Commands

```
option show_stats 1;

option solver "filter";
option solver "ipopt";
option solver "knitro";
option solver "loqo";

model;
include life.mod;

data;
include life.dat;

solve;

printf {t in 0..T}: "%21.15e  %21.15e\n", c[t], S[t] > cops.dat;
```

Part II

Continuous Optimization in AMPL

Modeling Languages

- Portable language for optimization problems
  - Algebraic description
  - Models easily modified and solved
  - Large problems can be processed
  - Programming language features
- Many available optimization algorithms
  - No need to compile C/FORTRAN code
  - Derivatives automatically calculated
  - Algorithms specific options can be set
- Communication with other tools
  - Relational databases and spreadsheets
  - MATLAB interface for function evaluations
- Excellent documentation
- Large user communities

Model Declaration

- Sets
  - Unordered, ordered, and circular sets
  - Cross products and point to set mappings
  - Set manipulation
- Parameters and variables
  - Initial and default values
  - Lower and upper bounds
  - Check statements
  - Defined variables
- Objective function and constraints
  - Equality, inequality, and range constraints
  - Complementarity constraints
  - Multiple objectives
- Problem statement

Data and Commands

- Data declaration
  - Set definitions
    - Explicit list of elements
    - Implicit list in parameter statements
  - Parameter definitions
    - Tables and transposed tables
    - Higher dimensional parameters
- Execution commands
  - Load model and data
  - Select problem, algorithm, and options
  - Solve the instance
  - Output results
- Other operations
  - Let and fix statements
  - Conditionals and loop constructs
  - Execution of external programs

Model Formulation

- Economy with n agents and m commodities
  - $e \in \Re^{n \times m}$ are the endowments
  - $\alpha \in \Re^{n \times m}$ and $\beta \in \Re^{n \times m}$ are the utility parameters
  - $\lambda \in \Re^{n}$ are the social weights
- Social planning problem

$$\begin{array}{lll} \max\limits_{x \geq 0} & \displaystyle\sum_{i=1}^{n} \lambda_i \left(\sum_{k=1}^{m} \frac{\alpha_{i,k}(1+x_{i,k})^{1-\beta_{i,k}}}{1-\beta_{i,k}} \right) & \\ \text{subject to} & \displaystyle\sum_{i=1}^{n} x_{i,k} \leq \sum_{i=1}^{n} e_{i,k} & \forall k = 1, \ldots, m \end{array}$$

Model: social1.mod

```
param n > 0, integer;                 # Agents
param m > 0, integer;                 # Commodities

param e {1..n, 1..m} >= 0, default 1; # Endowment

param lambda {1..n} > 0;              # Social weights
param alpha {1..n, 1..m} > 0;         # Utility parameters
param beta {1..n, 1..m} > 0;


var x{1..n, 1..m} >= 0;               # Consumption
var u{i in 1..n} =                    # Utility
  sum {k in 1..m} alpha[i,k] * (1 + x[i,k])^(1 - beta[i,k]) / (1 - beta[i,k]);


maximize welfare:
    sum {i in 1..n} lambda[i] * u[i];

subject to
  consumption {k in 1..m}:
    sum {i in 1..n} x[i,k] <= sum {i in 1..n} e[i,k];
```

Data: social1.dat

```
param n := 3;           # Agents
param m := 4;           # Commodities


param alpha : 1    2    3    4 :=
       1      1    1    1    1
       2      1    2    3    4
       3      2    1    1    5;

param beta (tr) : 1    2    3 :=
       1         1.5   2   0.6
       2         1.6   3   0.7
       3         1.7   2   2.0
       4         1.8   2   2.5;

param : lambda :=
    1     1
    2     1
    3     1;
```

Commands: social1.cmd

```
# Load model and data
model social1.mod;
data social1.dat;

# Specify solver and options
option solver "minos";
option minos_options "outlev=1";

# Solve the instance
solve;

# Output results
display x;
printf {i in 1..n} "%2d: % 5.4e\n", i, u[i];
```

Output

```
ampl: include social1.cmd;
MINOS 5.5: outlev=1
MINOS 5.5: optimal solution found.
25 iterations, objective 2.252422003
Nonlin evals: obj = 44, grad = 43.
x :=
1 1   0.0811471
1 2   0.574164
1 3   0.703454
1 4   0.267241
2 1   0.060263
2 2   0.604858
2 3   1.7239
2 4   1.47516
3 1   2.85859
3 2   1.82098
3 3   0.572645
3 4   1.2576
;

 1: -5.2111e+00
 2: -4.0488e+00
 3:  1.1512e+01
ampl: quit;
```

Model: social2.mod

```
set AGENTS;                                   # Agents
set COMMODITIES;                              # Commodities


param e {AGENTS, COMMODITIES} >= 0, default 1; # Endowment

param lambda {AGENTS} > 0;                    # Social weights
param alpha {AGENTS, COMMODITIES} > 0;        # Utility parameters
param beta {AGENTS, COMMODITIES} > 0;

param gamma {i in AGENTS, k in COMMODITIES} := 1 - beta[i,k];


var x{AGENTS, COMMODITIES} >= 0;              # Consumption
var u{i in AGENTS} =                          # Utility
  sum {k in COMMODITIES} alpha[i,k] * (1 + x[i,k])^gamma[i,k] / gamma[i,k];


maximize welfare:
    sum {i in AGENTS} lambda[i] * u[i];

subject to
  consumption {k in COMMODITIES}:
    sum {i in AGENTS} x[i,k] <= sum {i in AGENTS} e[i,k];
```

Data: social2.dat

```
set COMMODITIES := Books, Cars, Food, Pens;

param: AGENTS : lambda :=
       Jorge      1
       Sven       1
       Todd       1;


param alpha : Books   Cars   Food   Pens  :=
     Jorge       1      1      1      1
     Sven        1      2      3      4
     Todd        2      1      1      5;

param beta (tr): Jorge   Sven   Todd  :=
     Books          1.5     2      0.6
     Cars           1.6     3      0.7
     Food           1.7     2      2.0
     Pens           1.8     2      2.5;
```

Commands: social2.cmd

```
# Load model and data
model social2.mod;
data social2.dat;

# Specify solver and options
option solver "minos";
option minos_options "outlev=1";

# Solve the instance
solve;

# Output results
display x;
printf {i in AGENTS} "%5s: % 5.4e\n", i, u[i];
```

Output

```
ampl: include social2.cmd
MINOS 5.5: outlev=1
MINOS 5.5: optimal solution found.
25 iterations, objective 2.252422003
Nonlin evals: obj = 44, grad = 43.
x :=
Jorge Books   0.0811471
Jorge Cars    0.574164
Jorge Food    0.703454
Jorge Pens    0.267241
Sven  Books   0.060263
Sven  Cars    0.604858
Sven  Food    1.7239
Sven  Pens    1.47516
Todd  Books   2.85859
Todd  Cars    1.82098
Todd  Food    0.572645
Todd  Pens    1.2576
;

Jorge: -5.2111e+00
 Sven: -4.0488e+00
 Todd:  1.1512e+01
ampl: quit;
```

Model Formulation

- Route commodities through a network
  - $\mathcal{N}$ is the set of nodes
  - $\mathcal{A} \subseteq \mathcal{N} \times \mathcal{N}$ is the set of arcs
  - $\mathcal{K}$ is the set of commodities
  - α and β are the congestion parameters
  - b denotes the supply and demand
- Multicommodity network flow problem

$$\begin{array}{lll} \max\limits_{x \geq 0, f \geq 0} & \sum\limits_{(i,j)\in\mathcal{A}} \left(\alpha_{i,j} f_{i,j} + \beta_{i,j} f_{i,j}^4\right) & \\ \text{subject to} & \sum\limits_{(i,j)\in\mathcal{A}} x_{i,j,k} \leq \sum\limits_{(j,i)\in\mathcal{A}} x_{j,i,k} + b_{i,k} & \forall i \in \mathcal{N}, k \in \mathcal{K} \\ & f_{i,j} = \sum\limits_{k\in\mathcal{K}} x_{i,j,k} & \forall (i,j) \in \mathcal{A} \end{array}$$

Model: network.mod

```
set NODES;                                      # Nodes in network
set ARCS within NODES cross NODES;              # Arcs in network
set COMMODITIES := 1..3;                        # Commodities


param b {NODES, COMMODITIES} default 0;         # Supply/demand
check {k in COMMODITIES}:                       # Supply exceeds demand
  sum{i in NODES} b[i,k] >= 0;

param alpha{ARCS} >= 0;                         # Linear part
param beta{ARCS} >= 0;                          # Nonlinear part


var x{ARCS, COMMODITIES} >= 0;                  # Flow on arcs
var f{(i,j) in ARCS} =                          # Total flow
  sum {k in COMMODITIES} x[i,j,k];


minimize time:
  sum {(i,j) in ARCS} (alpha[i,j]*f[i,j] + beta[i,j]*f[i,j]^4);

subject to
  conserve {i in NODES, k in COMMODITIES}:
    sum {(i,j) in ARCS} x[i,j,k] <= sum{(j,i) in ARCS} x[j,i,k] + b[i,k];
```

Data: network.dat

```
set NODES := 1 2 3 4 5;

param: ARCS : alpha beta =
        1 2     1    0.5
        1 3     1    0.4
        2 3     2    0.7
        2 4     3    0.1
        3 2     1    0.0
        3 4     4    0.5
        4 1     5    0.0
        4 5     2    0.1
        5 2     0    1.0;


let b[1,1] :=  7;     # Node 1, Commodity 1 supply
let b[4,1] := -7;     # Node 4, Commodity 1 demand
let b[2,2] :=  3;     # Node 2, Commodity 2 supply
let b[5,2] := -3;     # Node 5, Commodity 2 demand
let b[3,3] :=  5;     # Node 1, Commodity 3 supply
let b[1,3] := -5;     # Node 4, Commodity 3 demand

fix {i in NODES, k in COMMODITIES: (i,i) in ARCS} x[i,i,k] := 0;
```

Commands: network.cmd

```
# Load model and data
model network.mod;
data network.dat;

# Specify solver and options
option solver "minos";
option minos_options "outlev=1";

# Solve the instance
solve;

# Output results
for {k in COMMODITIES} {
  printf "Commodity: %d\n", k > network.out;
  printf {(i,j) in ARCS: x[i,j,k] > 0} "%d.%d = % 5.4e\n", i, j, x[i,j,k] > network.out;
  printf "\n" > network.out;
}
```

Output

```
ampl: include network.cmd;
MINOS 5.5: outlev=1
MINOS 5.5: optimal solution found.
12 iterations, objective 1505.526478
Nonlin evals: obj = 14, grad = 13.
ampl: quit;
```

Results: network.out

```
Commodity: 1
1.2 =  3.3775e+00
1.3 =  3.6225e+00
2.4 =  6.4649e+00
3.2 =  3.0874e+00
3.4 =  5.3510e-01

Commodity: 2
2.4 =  3.0000e+00
4.5 =  3.0000e+00

Commodity: 3
3.4 =  5.0000e+00
4.1 =  5.0000e+00
```

Initial Coordinate Descent: wardrop0.cmd

```
# Load model and data
model network.mod;
data network.dat;

option solver "minos";
option minos_options "outlev=1";

# Coordinate descent method
fix {(i,j) in ARCS, k in COMMODITIES} x[i,j,k];
drop {i in NODES, k in COMMODITIES} conserve[i,k];

for {iter in 1..100} {
  for {k in COMMODITIES} {
    unfix {(i,j) in ARCS} x[i,j,k];
    restore {i in NODES} conserve[i,k];

    solve;

    fix {(i,j) in ARCS} x[i,j,k];
    drop {i in NODES} conserve[i,k];
  }
}

# Output results
for {k in COMMODITIES} {
  printf "\nCommodity: %d\n", k > network.out;
  printf {(i,j) in ARCS: x[i,j,k] > 0} "%d.%d = % 5.4e\n", i, j, x[i,j,k] > network.out;
}
```

Improved Coordinate Descent: wardrop.mod

```
set NODES;                                     # Nodes in network
set ARCS within NODES cross NODES;             # Arcs in network
set COMMODITIES := 1..3;                       # Commodities

param b {NODES, COMMODITIES} default 0;        # Supply/demand
param alpha {ARCS} >= 0;                       # Linear part
param beta {ARCS} >= 0;                        # Nonlinear part

var x {ARCS, COMMODITIES} >= 0;                # Flow on arcs
var f {(i,j) in ARCS} =                        # Total flow
  sum {k in COMMODITIES} x[i,j,k];

minimize time {k in COMMODITIES}:
  sum {(i,j) in ARCS} (alpha[i,j]*f[i,j] + beta[i,j]*f[i,j]^4);

subject to
  conserve {i in NODES, k in COMMODITIES}:
    sum {(i,j) in ARCS} x[i,j,k] <= sum{(j,i) in ARCS} x[j,i,k] + b[i,k];

problem subprob {k in COMMODITIES}: time[k], {i in NODES} conserve[i,k],
                                    {(i,j) in ARCS} x[i,j,k], f;
```

Improved Coordinate Descent: wardrop1.cmd

```
# Load model and data
model wardrop.mod;
data wardrop.dat;

# Specify solver and options
option solver "minos";
option minos_options "outlev=1";

# Coordinate descent method
for {iter in 1..100} {
  for {k in COMMODITIES} {
    solve subprob[k];
  }
}

for {k in COMMODITIES} {
  printf "Commodity: %d\n", k > wardrop.out;
  printf {(i,j) in ARCS: x[i,j,k] > 0} "%d.%d = % 5.4e\n", i, j, x[i,j,k] > wardrop.out;
  printf "\n" > wardrop.out;
}
```

Final Coordinate Descent: wardrop2.cmd

```
# Load model and data
model wardrop.mod;
data wardrop.dat;

# Specify solver and options
option solver "minos";
option minos_options "outlev=1";

# Coordinate descent method
param xold{ARCS, COMMODITIES};
param xnew{ARCS, COMMODITIES};

repeat {
  for {k in COMMODITIES} {
    problem subprob[k];
    let {(i,j) in ARCS} xold[i,j,k] := x[i,j,k];
    solve;
    let {(i,j) in ARCS} xnew[i,j,k] := x[i,j,k];
  }
} until (sum {(i,j) in ARCS, k in COMMODITIES} abs(xold[i,j,k] - xnew[i,j,k]) <= 1e-6);

for {k in COMMODITIES} {
  printf "Commodity: %d\n", k > wardrop.out;
  printf {(i,j) in ARCS: x[i,j,k] > 0} "%d.%d = % 5.4e\n", i, j, x[i,j,k] > wardrop.out;
  printf "\n" > wardrop.out;
}
```

Ordered Sets

```
param V, integer;                               # Number of vertices
param E, integer;                               # Number of elements

set VERTICES := {1..V};                         # Vertex indices
set ELEMENTS := {1..E};                         # Element indices
set COORDS   := {1..3} ordered;                 # Spatial coordinates

param T{ELEMENTS, 1..4} in VERTICES;            # Tetrahedral elements

var x{VERTICES, COORDS};                        # Position of vertices

var norm{e in ELEMENTS} = sum{i in COORDS, j in 1..4}
  (x[T[e,j], i] - x[T[e,1], i])^2;

var area{e in ELEMENTS} = sum{i in COORDS}
  (x[T[e,2], i] - x[T[e,1], i]) *
    ((x[T[e,3], nextw(i)] - x[T[e,1], nextw(i)]) *
     (x[T[e,4], prevw(i)] - x[T[e,1], prevw(i)]) -
     (x[T[e,3], prevw(i)] - x[T[e,1], prevw(i)]) *
     (x[T[e,4], nextw(i)] - x[T[e,1], nextw(i)]));

minimize f: sum {e in ELEMENTS} norm[e] / max(area[e], 0) ^ (2 / 3);
```

Circular Sets

```
param V, integer;                               # Number of vertices
param E, integer;                               # Number of elements

set VERTICES := {1..V};                         # Vertex indices
set ELEMENTS := {1..E};                         # Element indices
set COORDS   := {1..3} circular;                # Spatial coordinates

param T{ELEMENTS, 1..4} in VERTICES;            # Tetrahedral elements

var x{VERTICES, COORDS};                        # Position of vertices

var norm{e in ELEMENTS} = sum{i in COORDS, j in 1..4}
  (x[T[e,j], i] - x[T[e,1], i])^2;

var area{e in ELEMENTS} = sum{i in COORDS}
  (x[T[e,2], i] - x[T[e,1], i]) *
    ((x[T[e,3], next(i)] - x[T[e,1], next(i)]) *
     (x[T[e,4], prev(i)] - x[T[e,1], prev(i)]) -
     (x[T[e,3], prev(i)] - x[T[e,1], prev(i)]) *
     (x[T[e,4], next(i)] - x[T[e,1], next(i)]));

minimize f: sum {e in ELEMENTS} norm[e] / max(area[e], 0) ^ (2 / 3);
```

Part III

Optimization Software

Generic Nonlinear Optimization Problem

Nonlinear Programming (NLP) problem

$$\begin{cases} \underset{x}{\text{minimize}} & f(x) & \text{objective} \\ \text{subject to} & c(x) = 0 & \text{constraints} \\ & x \geq 0 & \text{variables} \end{cases}$$

- $f : R^n \to R$, $c : R^n \to R^m$ smooth (typically $\mathcal{C}^2$)
- $x \in R^n$ finite dimensional (may be large)
- more general $l \leq c(x) \leq u$ possible

Optimality Conditions for NLP

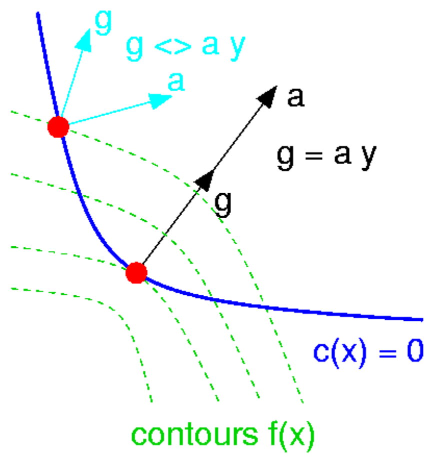
Constraint qualification (CQ)
Linearizations of $c(x) = 0$ characterize all feasible perturbations
$\Rightarrow$ rules out cusps etc.

x^* local minimizer & CQ holds $\Rightarrow \exists$ multipliers y^*, z^*:

$$\begin{aligned} \nabla f(x^*) - \nabla c(x^*)^T y^* - z^* &= 0 \\ c(x^*) &= 0 \\ X^* z^* &= 0 \\ x^* \geq 0,\ z^* &\geq 0 \end{aligned}$$

where $X^* = \text{diag}(x^*)$, thus $X^* z^* = 0 \Leftrightarrow x_i^* z_i^* = 0$
Lagrangian: $\mathcal{L}(x, y, z) := f(x) - y^T c(x) - z^T x$

Optimality Conditions for NLP



Objective gradient is linear combination of constraint gradients

$$g(x) = A(x)y, \qquad \text{where } g(x) := \nabla f(x),\ A(x) := \nabla c(x)^T$$

Newton's Method for Nonlinear Equations

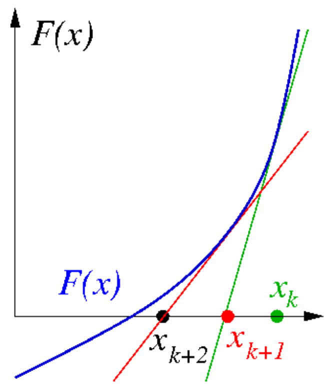
Solve $F(x) = 0$:
Get approx. x_{k+1} of solution of $F(x) = 0$
by solving linear model about x_k:

$$F(x_k) + \nabla F(x_k)^T (x - x_k) = 0$$

for $k = 0, 1, \ldots$

Theorem: If $F \in \mathcal{C}^2$, and $\nabla F(x^*)$ nonsingular,
then Newton converges quadratically near x^*.

Newton's Method for Nonlinear Equations



Next: two classes of methods based on Newton ...

Sequential Quadratic Programming (SQP)

Consider equality constrained NLP

$$\underset{x}{\text{minimize}} \; f(x) \quad \text{subject to } c(x) = 0$$

Optimality conditions:

$$\begin{aligned} \nabla f(x) - \nabla c(x)^T y &= 0 \quad \text{and} \\ c(x) &= 0 \end{aligned}$$

... system of nonlinear equations: $F(w) = 0$ for $w = (x, y)$.

... solve using Newton's method

Sequential Quadratic Programming (SQP)

Nonlinear system of equations (KKT conditions)

$$\nabla f(x) - \nabla c(x)^T y = 0 \quad \text{and} \qquad c(x) = 0$$

Apply Newton's method from $w_k = (x_k, y_k)$ ... $H_k = \nabla^2 \mathcal{L}(x_k, y_k)$

$$\begin{bmatrix} H_k & -A_k \\ A_k^T & 0 \end{bmatrix} \begin{pmatrix} s_x \\ s_y \end{pmatrix} = - \begin{pmatrix} \nabla_x \mathcal{L}(x_k, y_k) \\ c_k \end{pmatrix}$$

... set $(x_{k+1}, y_{k+1}) = (x_k + s_x, y_k + s_y)$ ... $A^k = \nabla c(x_k)^T$
... solve for $y_{k+1} = y_k + s_y$ directly instead:

$$\begin{bmatrix} H_k & -A_k \\ A_k^T & 0 \end{bmatrix} \begin{pmatrix} s \\ y_{k+1} \end{pmatrix} = - \begin{pmatrix} \nabla f_k \\ c_k \end{pmatrix}$$

... set $(x_{k+1}, y_{k+1}) = (x_k + s, y_{k+1})$

Sequential Quadratic Programming (SQP)

Newton's Method for KKT conditions leads to:

$$\begin{bmatrix} H_k & -A_k \\ A_k^T & 0 \end{bmatrix} \begin{pmatrix} s \\ y_{k+1} \end{pmatrix} = - \begin{pmatrix} \nabla f_k \\ c_k \end{pmatrix}$$

... are optimality conditions of QP

$$\begin{cases} \underset{s}{\text{minimize}} & \nabla f_k^T s + \frac{1}{2} s^T H_k s \\ \text{subject to} & c_k + A_k^T s = 0 \end{cases}$$

... hence Sequential Quadratic Programming

Sequential Quadratic Programming (SQP)

SQP for inequality constrained NLP:

$$\underset{x}{\text{minimize}} \;\; f(x) \quad \text{subject to } c(x) = 0 \quad \& \quad x \geq 0$$

REPEAT

1. Solve QP for (s, y_{k+1}, z_{k+1})

$$\begin{cases} \underset{s}{\text{minimize}} & \nabla f_k^T s + \frac{1}{2} s^T H_k s \\ \text{subject to} & c_k + A_k^T s = 0 \\ & x_k + s \geq 0 \end{cases}$$

2. Set $x_{k+1} = x_k + s$

Modern Interior Point Methods (IPM)

General NLP

$$\underset{x}{\text{minimize}} \;\; f(x) \quad \text{subject to } c(x) = 0 \quad \& \quad x \geq 0$$

Perturbed $\mu > 0$ optimality conditions $(x, z > 0)$

$$F_\mu(x, y, z) = \begin{Bmatrix} \nabla f(x) - \nabla c(x)^T y - z \\ c(x) \\ X z - \mu e \end{Bmatrix} = 0$$

- Primal-dual formulation, where $X = \text{diag}(x)$
- Central path $\{x(\mu), y(\mu), z(\mu) \; : \; \mu > 0\}$
- Apply Newton's method for sequence $\mu \searrow 0$

Modern Interior Point Methods (IPM)

Newton's method applied to primal-dual system ...

$$\begin{bmatrix} \nabla^2 \mathcal{L}_k & -A_k & -I \\ A_k^T & 0 & 0 \\ Z_k & 0 & X_k \end{bmatrix} \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix} = -F_\mu(x_k, y_k, z_k)$$

where $A_k = \nabla c(x_k)^T$, X_k diagonal matrix of x_k.

Polynomial run-time guarantee for convex problems

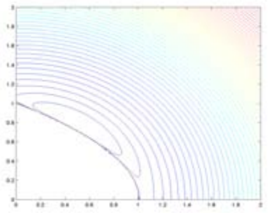
Classical Interior Point Methods (IPM)

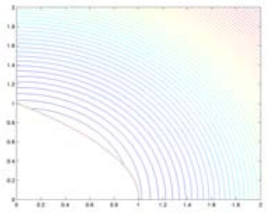
$$\underset{x}{\text{minimize}}\ f(x)\quad \text{subject to } c(x)=0 \quad \& \quad x\ge 0$$

Related to classical barrier methods [Fiacco & McCormick]

$$\begin{cases} \underset{x}{\text{minimize}} & f(x) - \mu \sum \log(x_i) \\ \text{subject to} & c(x)=0 \end{cases}$$

$\mu = 0.1$ $\mu = 0.001$





$$\text{minimize } x_1^2 + x_2^2 \quad \text{subject to} \quad x_1 + x_2^2 \ge 1$$

Classical Interior Point Methods (IPM)

$$\underset{x}{\text{minimize}}\ f(x)\quad \text{subject to } c(x)=0 \quad \& \quad x\ge 0$$

Relationship to barrier methods

$$\begin{cases} \underset{x}{\text{minimize}} & f(x) - \mu \sum \log(x_i) \\ \text{subject to} & c(x)=0 \end{cases}$$

First order conditions

$$\begin{aligned} \nabla f(x) - \mu X^{-1} e - A(x) y &= 0 \\ c(x) &= 0 \end{aligned}$$

... apply Newton's method ...

Classical Interior Point Methods (IPM)

Newton's method for barrier problem from x_k ...

$$\begin{bmatrix} \nabla^2 \mathcal{L}_k + \mu X_k^{-2} & -A_k \\ A_k^T & 0 \end{bmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \ldots$$

Introduce $Z(x_k) := \mu X_k^{-1}$ ... or ... $Z(x_k) X_k = \mu e$

$$\begin{bmatrix} \nabla^2 \mathcal{L}_k + Z(x_k) X_k^{-1} & -A_k \\ A_k & 0 \end{bmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \ldots$$

... compare to primal-dual system ...

Classical Interior Point Methods (IPM)

Recall: Newton's method applied to primal-dual system ...

$$\begin{bmatrix} \nabla^2 \mathcal{L}_k & -A_k & -I \\ A_k^T & 0 & 0 \\ Z_k & 0 & X_k \end{bmatrix} \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix} = -F_\mu(x_k, y_k, z_k)$$

Eliminate $\Delta z = -X^{-1} Z \Delta x - Ze - \mu X^{-1} e$

$$\begin{bmatrix} \nabla^2 \mathcal{L}_k + Z_k X_k^{-1} & -A_k \\ A_k & 0 \end{bmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \ldots$$

Interior Point Methods (IPM)

Primal-dual system ...

$$\begin{bmatrix} \nabla^2 \mathcal{L}_k + Z_k X_k^{-1} & -A_k \\ A_k & 0 \end{bmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \ldots$$

... compare to barrier system ...

$$\begin{bmatrix} \nabla^2 \mathcal{L}_k + Z(x_k) X_k^{-1} & -A_k \\ A_k & 0 \end{bmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \ldots$$

- Z_k is free, not $Z(x_k) = \mu X_k^{-1}$ (primal multiplier)
- avoid difficulties with barrier ill-conditioning

Convergence from Remote Starting Points

$$\underset{x}{\text{minimize}}\ f(x) \quad \text{subject to } c(x) = 0 \quad \& \quad x \geq 0$$

- Newton's method converges quadratically near a solution
- Newton's method may diverge if started far from solution
- How can we safeguard against this failure?

... motivates penalty or merit functions that

1. monitor progress towards a solution
2. combine objective $f(x)$ and constraint violation $\|c(x)\|$

Penalty Functions (i)

Augmented Lagrangian Methods

$$\underset{x}{\text{minimize}}\ L(x, y_k, \rho_k) \ = \ f(x) - y_k^T c(x) + \tfrac{1}{2}\rho_k \|c(x)\|^2$$

As $y_k \to y_*$: • $x_k \to x_*$ for $\rho_k > \bar{\rho}$
• No ill-conditioning, improves convergence rate

- update ρ_k based on reduction in $\|c(x)\|^2$
- approx. minimize $L(x, y_k, \rho_k)$
- first-order multiplier update: $y_{k+1} = y_k - \rho_k c(x_k)$
  $\Rightarrow$ dual iteration

Penalty Functions (ii)

Exact Penalty Function

$$\underset{x}{\text{minimize}}\ \Phi(x, \pi) \ = \ f(x) + \pi \|c(x)\|$$

- combine constraints ad objective
- equivalence of optimality $\Rightarrow$ exact for $\pi > \|y^*\|_D$
  ... now apply unconstrained techniques
- Φ nonsmooth, but equivalent to smooth problem (exercise)

... how do we enforce descent in merit functions???

Line Search Methods

SQP/IPM compute s descend direction: $s^T \nabla \Phi < 0$

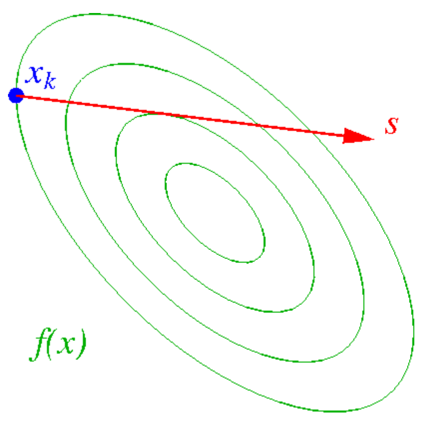
Backtracking-Armijo line search

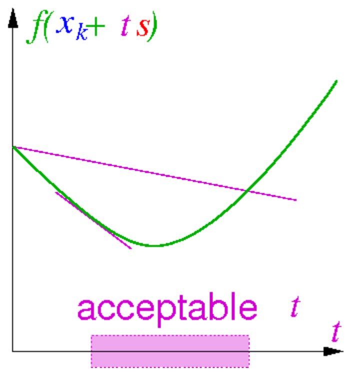
Given $\alpha^0 = 1$, $\beta = 0.1$, set $l = 0$

```
REPEAT
  1. $\alpha^{l+1} = \alpha^l / 2$ & evaluate $\Phi(x + \alpha^{l+1} s)$
  2. $l = l + 1$
UNTIL $\Phi(x + \alpha^l s) \le f(x) + \alpha^l \beta s^T \nabla \Phi$
```

Converges to stationary point, or unbounded, or zero descend

Line Search Methods

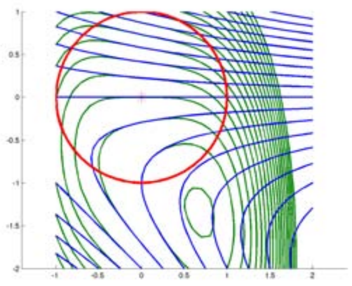


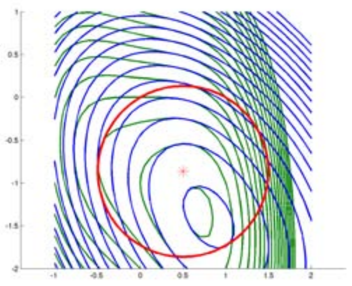


Trust Region Methods

Globalize SQP (IPM) by adding trust region, $\Delta^k > 0$

$$\left\{ \begin{array}{lll} \underset{s}{\text{minimize}} & \nabla f_k^T s + \frac{1}{2} s^T H_k s & \\ \text{subject to} & c_k + A_k^T s = 0, \quad x_k + s \ge 0, & \|s\| \le \Delta^k \end{array} \right.$$





Trust Region Methods

Globalize SQP (IPM) by adding trust region, $\Delta^k > 0$

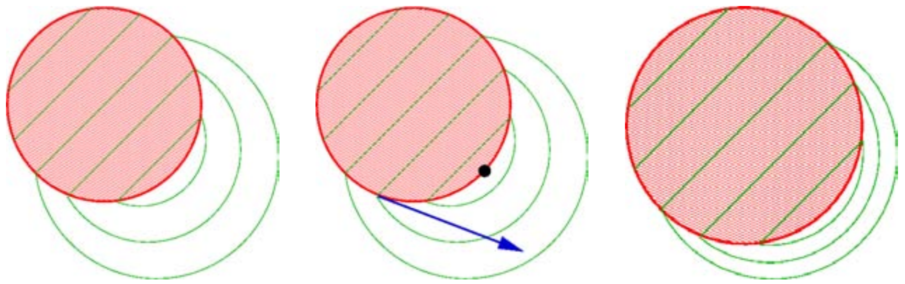
$$\left\{ \begin{array}{lll} \underset{s}{\text{minimize}} & \nabla f_k^T s + \frac{1}{2} s^T H_k s & \\ \text{subject to} & c_k + A_k^T s = 0, \quad x_k + s \ge 0, & \|s\| \le \Delta^k \end{array} \right.$$

```
REPEAT
  1. Solve QP approximation about $x_k$
  2. Compute actual/predicted reduction, $r_k$
  3. IF $r_k \ge 0.75$ THEN $x_{k+1} = x_k + s$                 increase $\Delta$
     ELSEIF $r_k \ge 0.25$ THEN $x_{k+1} = x_k + s$
     ELSE $x_{k+1} = x_k$ & decrease $\Delta$                   reject step
UNTIL convergence
```

Filter Methods for NLP

Penalty function can be inefficient

- Penalty parameter not known a priori
- Large penalty parameter $\Rightarrow$ slow convergence



Filter Methods for NLP

Penalty function can be inefficient

- Penalty parameter not known a priori
- Large penalty parameter $\Rightarrow$ slow convergence

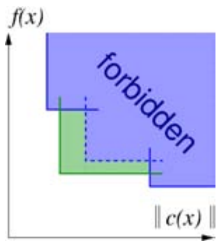
Two competing aims in optimization:

1. Minimize $f(x)$
2. Minimize $h(x) := \|c(x)\|$ ... more important

$\Rightarrow$ concept from multi-objective optimization:
(h_{k+1}, f_{k+1}) dominates (h_l, f_l) iff $h_{k+1} \le h_l$ & $f_{k+1} \le f_l$

Filter Methods for NLP

Filter $\mathcal{F}$: list of non-dominated pairs (h_l, f_l)



- new x_{k+1} acceptable to filter $\mathcal{F}$, iff
  1. $h_{k+1} \le h_l \; \forall l \in \mathcal{F}$, or
  2. $f_{k+1} \le f_l \; \forall l \in \mathcal{F}$
- remove redundant entries
- reject new x_{k+1},
  if $h_{k+1} > h_l$ & $f_{k+1} > f_l$
  ... reduce trust region radius $\Delta = \Delta/2$

$\Rightarrow$ often accept new x_{k+1}, even if penalty function increases

Sequential Quadratic Programming

- `filterSQP`
  - trust-region SQP; robust QP solver
  - filter to promote global convergence
- `SNOPT`
  - line-search SQP; null-space CG option
  - ℓ_1 exact penalty function
- `SLIQUE` (part of `KNITRO`)
  - SLP-EQP ("SQP" for larger problems)
  - trust-region with ℓ_1 penalty

Other Methods: `CONOPT` generalized reduced gradient method

Interior Point Methods

- IPOPT (free: part of COIN-OR)
  - line-search filter algorithm
  - 2nd order convergence analysis for filter
- KNITRO
  - trust-region Newton to solve barrier problem
  - ℓ_1 penalty barrier function
  - Newton system: direct solves or null-space CG
- LOQO
  - line-search method
  - Cholesky factorization; no convergence analysis

Other solvers: MOSEK (unsuitable or nonconvex problem)

Augmented Lagrangian Methods

- LANCELOT
  - minimize augmented Lagrangian subject to bounds
  - trust-region to force convergence
  - iterative (CG) solves
- MINOS
  - minimize augmented Lagrangian subject to linear constraints
  - line-search; recent convergence analysis
  - direct factorization of linear constraints
- PENNON
  - suitable for semi-definite optimization
  - alternative penalty terms

Automatic Differentiation

How do I get the derivatives $\nabla c(x)$, $\nabla^2 c(x)$ etc?

- hand-coded derivatives are error prone
- finite differences $\frac{\partial c_i(x)}{\partial x_j} \simeq \frac{c_i(x+\delta e_j)-c_i(x)}{\delta}$ can be dangerous
  where $e_j = (0,\ldots,0,1,0,\ldots,0)$ is j^{th} unit vector

Automatic Differentiation

- chain rule techniques to differentiate program
- recursive application $\Rightarrow$ "exact" derivatives
- suitable for huge problems, see www.autodiff.org

... already done for you in AMPL/GAMS etc.

Something Under the Bed is Drooling

1. floating point (IEEE) exceptions
2. unbounded problems
   2.1 unbounded objective
   2.2 unbounded multipliers
3. (locally) inconsistent problems
4. suboptimal solutions

SOMETHING UNDER THE BED IS DROOLING
A Calvin and Hobbes Collection by Bill Watterson
Foreword by Pat Oliphant

... identify problem & suggest remedies

Floating Point (IEEE) Exceptions

Bad example: minimize barrier function

```
param mu default 1;
var x{1..2} >= -10, <= 10;
var s;
minimize barrier: x[1]^2 + x[2]^2 - mu*log(s);
subject to
   cons: s = x[1] + x[2]^2 - 1;
```

... results in error message like

`Cannot evaluate objective at start`

... change initialization of s:

`var s := 1;` ... difficult, if IEEE during solve ...

Unbounded Objective

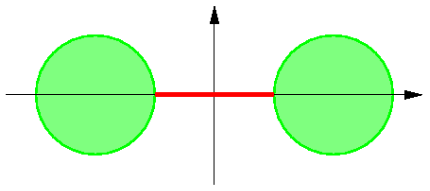
Penalized MPEC $\pi = 1$:

$$\begin{array}{ll} \underset{x}{\text{minimize}} & x_1^2 + x_2^2 - 4x_1x_2 \quad + \pi x_1 x_2 \\ \text{subject to} & x_1, x_2 \geq 0 \end{array}$$

... unbounded below for all $\pi < 2$

Locally Inconsistent Problems

NLP may have no feasible point



feasible set: intersection of circles

Locally Inconsistent Problems

NLP may have no feasible point

```
var x{1..2} >= -1;
minimize objf: -1000*x[2];
subject to
   con1: (x[1]+2)^2 + x[2]^2 <= 1;
   con2: (x[1]-2)^2 + x[2]^2 <= 1;
```

- not all solvers recognize this ...
- finding feasible point $\Leftrightarrow$ global optimization

Locally Inconsistent Problems

LOQO

```
     |          Primal           |            Dual
Iter |  Obj Value      Infeas    |   Obj Value      Infeas
- - - - - - - - - - - - - - - - - - - - - - - - - - - - -
  1   -1.000000e+03   4.2e+00    -6.000000e+00   1.0e-00
[...]
500    2.312535e-04   7.9e-01     1.715213e+12   1.5e-01
LOQO 6.06: iteration limit
```

... fails to converge ... not useful for user

dual unbounded $\to \infty \Rightarrow$ primal infeasible

Locally Inconsistent Problems

FILTER

```
iter  |    rho    |    ||d||    |    f / hJ   |    ||c||/hJt
------+-----------+-------------+-------------+------------
   0:0  10.0000     0.00000      -1000.0000     16.000000
   1:1  10.0000     2.00000      -1000.0000     8.0000000
[...]
   8:2  2.00000   0.320001E-02  7.9999693    0.10240052E-04
   9:2  2.00000   0.512000E-05  8.0000000    0.26214586E-10
filterSQP: Nonlinear constraints locally infeasible
```

... fast convergence to minimum infeasibility
... identify "blocking" constraints ... modify model/data

Locally Inconsistent Problems

Remedies for locally infeasible problems:

1. check your model: print constraints & residuals, e.g.
   ```
   solve;
   display _conname, _con.lb, _con.body, _con.ub;
   display _varname, _var.lb, _var, _var.ub;
   ```
   ... look at violated and active constraints
2. try different nonlinear solvers (easy with AMPL)
3. build-up model from few constraints at a time
4. try different starting points ... global optimization

Suboptimal Solution & Multi-start

Problems can have many local mimimizers

```
param pi := 3.1416;
param n integer, >= 0, default 2;
set N := 1..n;
var x{N} >= 0, <= 32*pi, := 1;
minimize objf:
- sum{i in N} x[i]*sin(sqrt(x[i]));
```

default start point converges to local minimizer

Suboptimal Solution & Multi-start

```
param nD := 5;        # discretization
set    D := 1..nD;
param  hD := 32*pi/(nD-1);
param optval{D,D};
model schwefel.mod;  # load model

for {i in D}{
   let x[1] := (i-1)*hD;
   for {j in D}{
      let x[2] := (j-1)*hD;
      solve;
      let optval[i,j] := objf;
   }; # end for
}; # end for
```

Suboptimal Solution & Multi-start

```
display optval;
optval [*,*]
:     1         2         3          4          5    :=
1     0         24.003   -36.29     -50.927     56.909
2   24.003     -7.8906   -67.580    -67.580    -67.580
3  -36.29     -67.5803  -127.27    -127.27    -127.27
4  -50.927    -67.5803  -127.27    -127.27    -127.27
5   56.909    -67.5803  -127.27    -127.27    -127.27
;
```

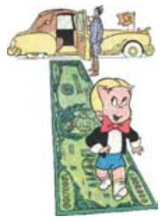
... there exist better multi-start procedures

Optimization with Integer Variables

- modeling discrete choices $\Rightarrow 0-1$ variables
- modeling integer decisions $\Rightarrow$ integer variables
  e.g. number of different stocks in portfolio (8-10)
  not number of beers sold at Goose Island (millions)

$\Rightarrow$ Mixed Integer Nonlinear Program (MINLP)
MINLP solvers:

- branch (separate $z_i = 0$ and $z_i = 1$) and cut
- solve millions of NLP relaxations: `MINLPBB, SBB`
- outer approximation: iterate MILP and NLP solvers
  `BONMIN` soon on `COIN-OR`



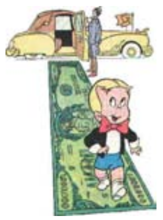
Portfolio Management

- N: Universe of asset to purchase
- x_i: Amount of asset i to hold
- B: Budget

$$\min_{x \in \mathbb{R}_+^{|N|}} \left\{ u(x) \mid \sum_{i \in N} x_i = B \right\}$$

- Markowitz: $u(x) \overset{\text{def}}{=} -\alpha^T x + \lambda x^T Q x$
  - α: Expected returns
  - Q: Variance-covariance matrix of expected returns
  - λ: Risk aversion parameter

More Realistic Models

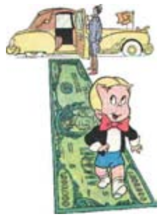


- $b \in \mathbb{R}^{|N|}$ of "benchmark" holdings
- Benchmark Tracking: $u(x) \overset{\text{def}}{=} (x-b)^T Q (x-b)$
  - Constraint on $\mathbb{E}$[Return]: $\alpha^T x \geq r$
- Limit Names: $|i \in N \ : \ x_i > 0| \leq K$
  - Use binary indicator variables to model the implication $x_i > 0 \Rightarrow y_i = 1$
  - Implication modeled with variable upper bounds:

$$x_i \leq B y_i \qquad \forall i \in N$$

  - $\sum_{i \in N} y_i \leq K$

Even More Models



- Min Holdings: $(x_i = 0) \vee (x_i \geq m)$
  - Model implication: $x_i > 0 \Rightarrow x_i \geq m$
  - $x_i > 0 \Rightarrow y_i = 1 \Rightarrow x_i \geq m$
  - $x_i \leq B y_i, x_i \geq m y_i \ \forall i \in N$
- Round Lots: $x_i \in \{k L_i, k = 1, 2, \dots\}$
  - $x_i - z_i L_i = 0, z_i \in \mathbb{Z}_+ \ \forall i \in N$
- Vector h of initial holdings
- Transactions: $t_i = |x_i - h_i|$
- Turnover: $\sum_{i \in N} t_i \leq \Delta$
- Transaction Costs: $\sum_{i \in N} c_i t_i$ in objective
- Market Impact: $\sum_{i \in N} \gamma_i t_i^2$ in objective

Global Optimization

I need to find the GLOBAL minimum!

- use any NLP solver (often work well!)
- use the multi-start trick from previous slides
- global optimization based on branch-and-reduce: `BARON`
  - constructs global underestimators
  - refines region by branching
  - tightens bounds by solving LPs
  - solve problems with 100s of variables
- "voodoo" solvers: genetic algorithm & simulated annealing
  no convergence theory ... usually worse than deterministic

Derivative-Free Optimization

My model does not have derivatives!

- Change your model ... good models have derivatives!
- pattern-search methods for min $f(x)$
  - evaluate $f(x)$ at stencil $x_k + \Delta M$
  - move to new best point
  - extend to NLP; some convergence theory
  - matlab: `NOMADm.m`; parallel `APPSPACK`
- solvers based on building quadratic models
- "voodoo" solvers: genetic algorithm & simulated annealing
  no convergence theory ... usually worse than deterministic

COIN-OR

`http://www.coin-or.org`

- COmputational INfrastructure for Operations Research
- A library of (interoperable) software tools for optimization
- A development platform for open source projects in the OR community
- Possibly Relevant Modules:
  - OSI: Open Solver Interface
  - CGL: Cut Generation Library
  - CLP: Coin Linear Programming Toolkit
  - CBC: Coin Branch and Cut
  - IPOPT: Interior Point OPTimizer for NLP
  - NLPAPI: NonLinear Programming API

Part IV

Complementarity Problems in AMPL

Definition

- Non-cooperative game played by n individuals
  - Each player selects a strategy to optimize their objective
  - Strategies for the other players are fixed
- Equilibrium reached when no improvement is possible
- Characterization of two player equilibrium (x^*, y^*)

$$x^* \in \left\{ \begin{array}{ll} \displaystyle\operatorname*{arg\,min}_{x \geq 0} & f_1(x, y^*) \\ \text{subject to} & c_1(x) \leq 0 \end{array} \right.$$

$$y^* \in \left\{ \begin{array}{ll} \displaystyle\operatorname*{arg\,min}_{y \geq 0} & f_2(x^*, y) \\ \text{subject to} & c_2(y) \leq 0 \end{array} \right.$$

- Many applications in economics
  - Bi-matrix games
  - Cournot duopoly models
  - General equilibrium models
  - Arrow-Debreau models

Complementarity Formulation

- Assume each optimization problem is convex
  - $f_1(\cdot, y)$ is convex for each y
  - $f_2(x, \cdot)$ is convex for each x
  - $c_1(\cdot)$ and $c_2(\cdot)$ satisfy constraint qualification
- Then the first-order conditions are necessary and sufficient

$$\begin{array}{ll} \displaystyle\min_{x \geq 0} & f_1(x, y^*) \\ \text{subject to} & c_1(x) \leq 0 \end{array} \quad \Leftrightarrow \quad \begin{array}{lcl} 0 \leq x & \perp & \nabla_x f_1(x, y^*) + \lambda_1^T \nabla_x c_1(x) \geq 0 \\ 0 \leq \lambda_1 & \perp & -c_1(x) \geq 0 \end{array}$$

$$\begin{array}{ll} \displaystyle\min_{y \geq 0} & f_2(x^*, y) \\ \text{subject to} & c_2(y) \leq 0 \end{array} \quad \Leftrightarrow \quad \begin{array}{lcl} 0 \leq y & \perp & \nabla_y f_2(x^*, y) + \lambda_2^T \nabla_y c_2(y) \geq 0 \\ 0 \leq \lambda_2 & \perp & -c_2(y) \geq 0 \end{array}$$

Formulation

- Firm $f \in \mathcal{F}$ chooses output x_f to maximize profit
  - u is the utility function

$$u = \left(1 + \sum_{f \in \mathcal{F}} x_f^\alpha\right)^{\frac{\eta}{\alpha}}$$

  - α and η are parameters
  - c_f is the unit cost for each firm
- In particular, for each firm $f \in \mathcal{F}$

$$x_f^* \quad \in \quad \operatorname*{arg\,max}_{x_f \geq 0} \left(\frac{\partial u}{\partial x_f} - c_f\right) x_f$$

- First-order optimality conditions

$$0 \leq x_f \quad \perp c_f - \frac{\partial u}{\partial x_f} - x_f \frac{\partial^2 u}{\partial x_f^2} \geq 0$$

Model: oligopoly.mod

```
set FIRMS;                                      # Firms in problem

param c {FIRMS};                                # Unit cost
param alpha > 0;                                # Constants
param eta > 0;

var x {FIRMS} default 0.1;                      # Output (no bounds!)

var s = 1 + sum {f in FIRMS} x[f]^alpha;        # Summation term
var u = s^(eta/alpha);                          # Utility
var du {f in FIRMS} =                           # Marginal price
  eta * s^(eta/alpha - 1) * x[f]^(alpha - 1);
var dudu {f in FIRMS} =                         # Derivative
  eta * (eta - alpha) * s^(eta/alpha - 2) * x[f]^(2 * alpha - 2) +
  eta * (alpha - 1  ) * s^(eta/alpha - 1) * x[f]^(    alpha - 2);

compl {f in FIRMS}:
  0 <= x[f] complements c[f] - du[f] - x[f] * dudu[f] >= 0;
```

Data: oligopoly.dat

```
param: FIRMS :   c :=
         1      0.07
         2      0.08
         3      0.09;

param alpha := 0.999;
param eta := 0.2;
```

Commands: oligopoly.cmd

```
# Load model and data
model oligopoly.mod;
data oligopoly.dat;

# Specify solver and options
option presolve 0;
option solver "pathampl";

# Solve complementarity problem
solve;

# Output the results
printf {f in FIRMS} "Output for firm %2d: % 5.4e\n", f, x[f] > oligcomp.out;
```

Results: oligopoly.out

```
Output for firm  1:  8.3735e-01
Output for firm  2:  5.0720e-01
Output for firm  3:  1.7921e-01
```

Model Formulation

- Economy with n agents and m commodities
  - $e \in \Re^{n \times m}$ are the endowments
  - $\alpha \in \Re^{n \times m}$ and $\beta \in \Re^{n \times m}$ are the utility parameters
  - $p \in \Re^{m}$ are the commodity prices
- Agent i maximizes utility with budget constraint

$$\begin{array}{ll} \max\limits_{x_{i,*} \geq 0} & \displaystyle\sum_{k=1}^{m} \frac{\alpha_{i,k}(1+x_{i,k})^{1-\beta_{i,k}}}{1-\beta_{i,k}} \\ \text{subject to} & \displaystyle\sum_{k=1}^{m} p_k\left(x_{i,k}-e_{i,k}\right) \leq 0 \end{array}$$

- Market k sets price for the commodity

$$0 \leq p_k \quad \perp \quad \sum_{i=1}^{n}(e_{i,k}-x_{i,k}) \geq 0$$

Model: cge.mod

```
set AGENTS;                                     # Agents
set COMMODITIES;                                # Commodities

param e {AGENTS, COMMODITIES} >= 0, default 1; # Endowment

param alpha {AGENTS, COMMODITIES} > 0;          # Utility parameters
param beta {AGENTS, COMMODITIES} > 0;

var x {AGENTS, COMMODITIES};                    # Consumption (no bounds!)
var l {AGENTS};                                 # Multipliers (no bounds!)
var p {COMMODITIES};                            # Prices (no bounds!)

var du {i in AGENTS, k in COMMODITIES} =        # Marginal prices
  alpha[i,k] / (1 + x[i,k])^beta[i,k];

subject to
  optimality {i in AGENTS, k in COMMODITIES}:
    0 <= x[i,k] complements -du[i,k] + p[k] * l[i] >= 0;

  budget {i in AGENTS}:
    0 <= l[i]    complements sum {k in COMMODITIES} p[k]*(e[i,k] - x[i,k]) >= 0;

  market {k in COMMODITIES}:
    0 <= p[k]    complements sum {i in AGENTS} (e[i,k] - x[i,k]) >= 0;
```

Data: cge.dat

```
set AGENTS := Jorge, Sven, Todd;
set COMMODITIES := Books, Cars, Food, Pens;

param alpha : Books  Cars  Food  Pens :=
     Jorge      1     1     1     1
     Sven       1     2     3     4
     Todd       2     1     1     5;

param beta (tr): Jorge  Sven  Todd :=
     Books        1.5    2     0.6
     Cars         1.6    3     0.7
     Food         1.7    2     2.0
     Pens         1.8    2     2.5;
```

Commands: cge.cmd

```
# Load model and data
model cge.mod;
data cge.dat;

# Specify solver and options
option presolve 0;
option solver "pathampl";

# Solve the instance
solve;

# Output results
printf {i in AGENTS, k in COMMODITIES} "%5s %5s: % 5.4e\n", i, k, x[i,k] > cge.out;
printf "\n" > cge.out;
printf {k in COMMODITIES} "%5s: % 5.4e\n", k, p[k] > cge.out;
```

Results: cge.out

```
Jorge Books:  8.9825e-01
Jorge  Cars:  1.4651e+00
Jorge  Food:  1.2021e+00
Jorge  Pens:  6.8392e-01
 Sven Books:  2.5392e-01
 Sven  Cars:  7.2054e-01
 Sven  Food:  1.6271e+00
 Sven  Pens:  1.4787e+00
 Todd Books:  1.8478e+00
 Todd  Cars:  8.1431e-01
 Todd  Food:  1.7081e-01
 Todd  Pens:  8.3738e-01

Books:  1.0825e+01
 Cars:  6.6835e+00
 Food:  7.3983e+00
 Pens:  1.1081e+01
```

Commands: cgenum.cmd

```
# Load model and data
model cge.mod;
data cge.dat;

# Specify solver and options
option presolve 0;
option solver "pathampl";

# Solve the instance
drop market['Books'];
fix p['Books'] := 1;
solve;

# Output results
printf {i in AGENTS, k in COMMODITIES} "%5s %5s: % 5.4e\n", i, k, x[i,k] > cgenum.out;
printf "\n" > cgenum.out;
printf {k in COMMODITIES} "%5s: % 5.4e\n", k, p[k] > cgenum.out;
```

Results: cgenum.out

```
Jorge Books:  8.9825e-01
Jorge  Cars:  1.4651e+00
Jorge  Food:  1.2021e+00
Jorge  Pens:  6.8392e-01
 Sven Books:  2.5392e-01
 Sven  Cars:  7.2054e-01
 Sven  Food:  1.6271e+00
 Sven  Pens:  1.4787e+00
 Todd Books:  1.8478e+00
 Todd  Cars:  8.1431e-01
 Todd  Food:  1.7081e-01
 Todd  Pens:  8.3738e-01

Books:  1.0000e+00
 Cars:  6.1742e-01
 Food:  6.8345e-01
 Pens:  1.0237e+00
```

Formulation

- Players select strategies to minimize loss
  - $p \in \Re^n$ is the probability player 1 chooses each strategy
  - $q \in \Re^m$ is the probability player 2 chooses each strategy
  - $A \in \Re^{n \times m}$ is the loss matrix for player 1
  - $B \in \Re^{n \times m}$ is the loss matrix for player 2
- Optimization problem for player 1

$$\begin{array}{ll} \displaystyle\min_{0 \le p \le 1} & p^T A q \\ \text{subject to} & e^T p = 1 \end{array}$$

- Optimization problem for player 2

$$\begin{array}{ll} \displaystyle\min_{0 \le q \le 1} & p^T B q \\ \text{subject to} & e^T q = 1 \end{array}$$

- Complementarity problem

$$0 \le p \le 1 \quad \perp \quad Aq - \lambda_1$$

Model: bimatrix1.mod

```
param n > 0, integer;            # Strategies for player 1
param m > 0, integer;            # Strategies for player 2

param A{1..n, 1..m};             # Loss matrix for player 1
param B{1..n, 1..m};             # Loss matrix for player 2

var p{1..n};                     # Probability player 1 selects strategy i
var q{1..m};                     # Probability player 2 selects strategy j
var lambda1;                     # Multiplier for constraint
var lambda2;                     # Multiplier for constraint

subject to
  opt1 {i in 1..n}:              # Optimality conditions for player 1
    0 <= p[i] <= 1 complements sum{j in 1..m} A[i,j] * q[j] - lambda1;

  opt2 {j in 1..m}:              # Optimality conditions for player 2
    0 <= q[j] <= 1 complements sum{i in 1..n} B[i,j] * p[i] - lambda2;

  con1:
    lambda1 complements sum{i in 1..n} p[i] = 1;

  con2:
    lambda2 complements sum{j in 1..m} q[j] = 1;
```

Model: bimatrix2.mod

```
param n > 0, integer;            # Strategies for player 1
param m > 0, integer;            # Strategies for player 2

param A{1..n, 1..m};             # Loss matrix for player 1
param B{1..n, 1..m};             # Loss matrix for player 2

var p{1..n};                     # Probability player 1 selects strategy i
var q{1..m};                     # Probability player 2 selects strategy j
var lambda1;                     # Multiplier for constraint
var lambda2;                     # Multiplier for constraint

subject to
  opt1 {i in 1..n}:              # Optimality conditions for player 1
    0 <= p[i] complements sum{j in 1..m} A[i,j] * q[j] - lambda1 >= 0;

  opt2 {j in 1..m}:              # Optimality conditions for player 2
    0 <= q[j] complements sum{i in 1..n} B[i,j] * p[i] - lambda2 >= 0;

  con1:
    0 <= lambda1 complements sum{i in 1..n} p[i] >= 1;

  con2:
    0 <= lambda2 complements sum{j in 1..m} q[j] >= 1;
```

Model: bimatrix3.mod

```
param n > 0, integer;            # Strategies for player 1
param m > 0, integer;            # Strategies for player 2

param A{1..n, 1..m};             # Loss matrix for player 1
param B{1..n, 1..m};             # Loss matrix for player 2

var p{1..n};                     # Probability player 1 selects strategy i
var q{1..m};                     # Probability player 2 selects strategy j

subject to
  opt1 {i in 1..n}:              # Optimality conditions for player 1
    0 <= p[i] complements sum{j in 1..m} A[i,j] * q[j] >= 1;

  opt2 {j in 1..m}:              # Optimality conditions for player 2
    0 <= q[j] complements sum{i in 1..n} B[i,j] * p[i] >= 1;
```

Pitfalls

- Nonsquare systems
  - Side variables
  - Side constraints
- Orientation of equations
  - Skew symmetry preferred
  - Proximal point perturbation
- AMPL presolve
  - `option presolve 0;`

Definition

- Leader-follower game
  - Dominant player (leader) selects a strategy y^*
  - Then followers respond by playing a Nash game

$$x_i^* \in \left\{ \begin{array}{ll} \underset{x_i \geq 0}{\text{arg min}} & f_i(x,y) \\ \text{subject to} & c_i(x_i) \leq 0 \end{array} \right.$$

- Leader solves optimization problem with equilibrium constraints

$$\begin{array}{lll} \underset{y \geq 0, x, \lambda}{\min} & g(x,y) & \\ \text{subject to} & h(y) \leq 0 & \\ & 0 \leq x_i \quad \perp \quad \nabla_{x_i} f_i(x,y) + \lambda_i^T \nabla_{x_i} c_i(x_i) \geq 0 \\ & 0 \leq \lambda_i \quad \perp \quad -c_i(x_i) \geq 0 \end{array}$$

- Many applications in economics
  - Optimal taxation
  - Tolling problems

Nonlinear Programming Formulation

$$\begin{array}{ll} \underset{x,y,\lambda,s,t \geq 0}{\min} & g(x,y) \\ \text{subject to} & h(y) \leq 0 \\ & s_i = \nabla_{x_i} f_i(x,y) + \lambda_i^T \nabla_{x_i} c_i(x_i) \\ & t_i = -c_i(x_i) \\ & \sum_i \left(s_i^T x_i + \lambda_i t_i\right) \leq 0 \end{array}$$

- Constraint qualification fails
  - Lagrange multiplier set unbounded
  - Constraint gradients linearly dependent
  - Central path does not exist
- Able to prove convergence results for some methods
- Reformulation very successful and versatile in practice

Penalization Approach

$$\begin{array}{ll} \underset{x,y,\lambda,s,t \geq 0}{\min} & g(x,y) + \pi \sum_i \left(s_i^T x_i + \lambda_i t_i\right) \\ \text{subject to} & h(y) \leq 0 \\ & s_i = \nabla_{x_i} f_i(x,y) + \lambda_i^T \nabla_{x_i} c_i(x_i) \\ & t_i = -c_i(x_i) \end{array}$$

- Optimization problem satisfies constraint qualification
- Need to increase π

Relaxation Approach

$$\begin{array}{ll} \min\limits_{x,y,\lambda,s,t\geq 0} & g(x,y) \\ \text{subject to} & h(y) \leq 0 \\ & s_i = \nabla_{x_i} f_i(x,y) + \lambda_i^T \nabla_{x_i} c_i(x_i) \\ & t_i = -c_i(x_i) \\ & \sum\limits_i \left(s_i^T x_i + \lambda_i t_i \right) \leq \tau \end{array}$$

- Need to decrease τ

Model Formulation

- Economy with n agents and m commodities
  - $e \in \Re^{n\times m}$ are the endowments
  - $\alpha \in \Re^{n\times m}$ and $\beta \in \Re^{n\times m}$ are the utility parameters
  - $p \in \Re^{m}$ are the commodity prices
- Agent i maximizes utility with budget constraint

$$\begin{array}{ll} \max\limits_{x_{i,*}\geq 0} & \sum\limits_{k=1}^{m} \frac{\alpha_{i,k}(1+x_{i,k})^{1-\beta_{i,k}}}{1-\beta_{i,k}} \\ \text{subject to} & \sum\limits_{k=1}^{m} p_k \left(x_{i,k} - e_{i,k}\right) \leq 0 \end{array}$$

- Market k sets price for the commodity

$$0 \leq p_k \quad \perp \quad \sum_{i=1}^{n} \left(e_{i,k} - x_{i,k}\right) \geq 0$$

Model: cgempec.mod

```
set LEADER;                                    # Leader
set FOLLOWERS;                                 # Followers
set AGENTS := LEADER union FOLLOWERS;          # All the agents
check: (card(LEADER) == 1 && card(LEADER inter FOLLOWERS) == 0);

set COMMODITIES;                               # Commodities

param e {AGENTS, COMMODITIES} >= 0, default 1; # Endowment

param alpha {AGENTS, COMMODITIES} > 0;         # Utility parameters
param beta {AGENTS, COMMODITIES} > 0;

var x {AGENTS, COMMODITIES};                   # Consumption (no bounds!)
var l {FOLLOWERS};                             # Multipliers (no bounds!)
var p {COMMODITIES};                           # Prices (no bounds!)

var u {i in AGENTS} =                          # Utility
  sum {k in COMMODITIES} alpha[i,k] * (1 + x[i,k])^(1 - beta[i,k]) / (1 - beta[i,k]);
var du {i in AGENTS, k in COMMODITIES} =       # Marginal prices
  alpha[i,k] / (1 + x[i,k])^beta[i,k];
```

Model: cgempec.mod

```
maximize
  objective: sum {i in LEADER} u[i];

subject to
  leader_budget {i in LEADER}:
    sum {k in COMMODITIES} p[k]*(e[i,k] - x[i,k]) >= 0;

  optimality {i in FOLLOWERS, k in COMMODITIES}:
    0 <= x[i,k] complements -du[i,k] + p[k] * l[i] >= 0;

  budget {i in FOLLOWERS}:
    0 <= l[i]    complements sum {k in COMMODITIES} p[k]*(e[i,k] - x[i,k]) >= 0;

  market {k in COMMODITIES}:
    0 <= p[k]    complements sum {i in AGENTS} (e[i,k] - x[i,k]) >= 0;
```

Data: cgempec.dat

```
set LEADER := Jorge;
set FOLLOWERS := Sven, Todd;
set COMMODITIES := Books, Cars, Food, Pens;

param alpha : Books  Cars  Food  Pens  :=
     Jorge      1     1     1     1
     Sven       1     2     3     4
     Todd       2     1     1     5;

param beta (tr): Jorge  Sven  Todd  :=
     Books         1.5    2     0.6
     Cars          1.6    3     0.7
     Food          1.7    2     2.0
     Pens          1.8    2     2.5;
```

Commands: cgempec.cmd

```
# Load model and data
model cgempec.mod;
data cgempec.dat;

# Specify solver and options
option presolve 0;
option solver "loqo";

# Solve the instance
drop market['Books'];
fix p['Books'] := 1;
solve;

# Output results
printf {i in AGENTS, k in COMMODITIES} "%5s %5s: % 5.4e\n", i, k, x[i,k] > cgempec.out;
printf "\n" > cgempec.out;
printf {k in COMMODITIES} "%5s: % 5.4e\n", k, p[k] > cgempec.out;
```

Output: cgempec.out

```
     Stackleberg                     Nash Game
Jorge Books:  9.2452e-01        Jorge Books:  8.9825e-01
Jorge  Cars:  1.3666e+00        Jorge  Cars:  1.4651e+00
Jorge  Food:  1.1508e+00        Jorge  Food:  1.2021e+00
Jorge  Pens:  7.7259e-01        Jorge  Pens:  6.8392e-01
 Sven Books:  2.5499e-01         Sven Books:  2.5392e-01
 Sven  Cars:  7.4173e-01         Sven  Cars:  7.2054e-01
 Sven  Food:  1.6657e+00         Sven  Food:  1.6271e+00
 Sven  Pens:  1.4265e+00         Sven  Pens:  1.4787e+00
 Todd Books:  1.8205e+00         Todd Books:  1.8478e+00
 Todd  Cars:  8.9169e-01         Todd  Cars:  8.1431e-01
 Todd  Food:  1.8355e-01         Todd  Food:  1.7081e-01
 Todd  Pens:  8.0093e-01         Todd  Pens:  8.3738e-01

Books:  1.0000e+00              Books:  1.0000e+00
 Cars:  5.9617e-01               Cars:  6.1742e-01
 Food:  6.6496e-01               Food:  6.8345e-01
 Pens:  1.0700e+00               Pens:  1.0237e+00
```

Unbounded Multipliers

```
var z{1..2} >= 0;
var z3;

minimize objf: z[1] + z[2] - z3;
subject to
  lin1:   -4 * z[1] + z3 <= 0;
  lin2:   -4 * z[2] + z3 <= 0;
  compl:  z[1]*z[2] <= 0;
```

LOQO Output

```
LOQO 6.06: outlev=2
     |          Primal         |            Dual
Iter |  Obj Value    Infeas    |  Obj Value   Infeas
- - - - - - - - - - - - - - - - - - - - - - - - - -
  1     1.000000e+00 0.0e+00    0.000000e+00 1.1e+00
  2     6.902180e-01 2.2e-01  -2.672676e-01 2.6e-01
  3     2.773222e-01 1.6e-01  -3.051049e-01 1.1e-01

292   -8.213292e-05 1.7e-09  -4.106638e-05 9.1e-07
293   -8.202525e-05 1.7e-09  -4.101255e-05 9.1e-07

294             nan    nan            nan      nan
500             nan    nan            nan      nan
```

FILTER Output

```
iter  |     rho     |    ||d||    |    f / hJ    |    ||c||/hJt
------+-------------+-------------+--------------+--------------
   0:0  10.0000       0.00000      1.0000000       0.0000000
   1:1  10.0000       1.00000      0.0000000       0.0000000

  24:1 0.156250      0.196695E-05-0.98347664E-06 0.24180657E-12
  25:1 0.156250      0.983477E-06-0.49173832E-06 0.60451644E-13

Norm of KKT residual................  0.471404521
max( |lam_i| * || a_i ||)...........  2.06155281
Largest modulus multiplier..........  2711469.25
```

Limitations

- Multipliers may not exist
- Solvers can have a hard time computing solutions
  - Try different algorithms
  - Compute feasible starting point
- Stationary points may have descent directions
  - Checking for descent is an exponential problem
  - Strong stationary points found in certain cases
- Many stationary points – global optimization
- Formulation of follower problem
  - Multiple solutions to Nash game
  - Nonconvex objective or constraints
  - Existence of multipliers