Learning &
Forgetting

Mark
Satterthwaite

Introduction

Model

Representation of
Results

Homotopy
Approach to
Equilibrium
Multiplicity

Tracing Out the
Full Equilibrium
Correspondence

Summary

Learning-by-Doing, Organizational Forgetting, and Industry Dynamics: Computational Issues

This presentation is based on David Besanko, Ulrich Doraszelski, Yaroslav Kryukov, and Mark A. Satterthwaite, "Learning-by-Doing, Organizational Forgetting, and Industry Dynamics," February 8, 2007.

http://www.kellogg.northwestern.edu/faculty/satterthwaite/research/2 0227%20MPE%20exist%20

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Learning-by-Doing and Organizational Forgetting

- Learning-by-Doing (LBD)
  The out-of-pocket cost for Big Boat Ship Building to construct a new, series X barge is:

Barge #	1	2	3	4	5	6
Cost	20	18	16	14	12	10.5

  Big Boat has signed, fixed price contracts to deliver two barges in 2008 and two in 2010. A buyer approaches the yard and offers $15 for delivery of a barge in 2009. Should Big Boat accept?

  Spence (*Bell Journal of Econ. & Management Sci.,* 1981), Cabral & Riordan (*Econometrica, 1994*)

- Organization Forgetting (OF)

  Argote, Beckman, & Epple (*Management Science,*1990), Benkard (*AER, 2000*)

- Benkard (*Rand Journal,* 2004) showed within an analysis of the early competition between Boeing, Douglas, and Lockheed in widebody passenger jets that LBD and OF together can explain Lockheed's disastrous history with its L1011.

Plan of Talk

- Goal of paper: What is the variety of equilibria that can occur in a duopoly in the presence of LBD and OF? We use Ericson and Pakes' computational Markov perfect equilibrium model to explore this question.

- Representation of results

- Homotopy approach to equilibrium multiplicity
  - Impossibility of Pakes-McGuire algorithm to identify more than a fraction of multiple equilibria

- Tracing out the equilibrium correspondence
  - Wealth of equilibria appear to exist.

Setup and Timing

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Basic Structure
Equilibrium
Paramerterization

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Results

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Equilibrium
Multiplicity

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Full Equilibrium
Correspondence

Summary

- Discrete time, infinite horizon.
- Two firms with potentially different stocks of know-how $(e_1, e_2) \in \{1, \ldots, M\}^2$.
- In each period, the timing is as follows:
  - Firms choose prices.
  - One buyer enters the market and makes a purchase.
  - Learning-by-doing and organizational forgetting occur and the firms' stocks of know-how change accordingly.
- Law of motion:
$$e'_n = e_n + q_n - f_n,$$
where
  - $q_n \in \{0, 1\}$ indicates whether firm n makes a sale;
  - $f_n \in \{0, 1\}$ represents organizational forgetting.

Learning-by-Doing

- Marginal cost of production:

$$c(e_n) = \begin{cases} \kappa e_n^{\eta} & \text{if} \quad 1 \leq e_n < m, \\ \kappa m^{\eta} & \text{if} \quad m \leq e_n \leq M, \end{cases}$$

where

- $\eta = \log_2 \rho$ for a progress ratio of $\rho \in (0, 1]$;
- κ is marginal cost at top of learning curve;
- m is bottom of learning curve.

- Marginal cost decreases by $1 - \rho$ percent as the stock of know-how doubles.

Organizational Forgetting

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Equilibrium
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Summary

- Probability of losing a unit of know-how:

$$\Pr(f_n = 1) = \Delta(e_n) = 1 - (1 - \delta)^{e_n},$$

  where $\delta \in [0, 1]$ is the forgetting rate.

- $\Delta(e_n)$ is increasing in e_n in line with
  - experimental evidence in management literature;
  - Jost's second law in psychology literature;
  - capital-stock model.

- Cabral & Riordan (1994) analyze the special case of $\delta = 0$.

Demand

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Model

Basic Structure

Equilibrium
Paramerterization

Representation of
Results

Homotopy
Approach to
Equilibrium
Multiplicity

Tracing Out the
Full Equilibrium
Correspondence

Summary

- In each period, one buyer enters the market and makes a purchase.
- A buyer's idiosyncratic preferences are unobservable to firms.
- Demand is logit. Thus probability of making a sale is:

$$\Pr(q_n = 1) = D_n(p_1, p_2) = \frac{1}{1 + \exp(\frac{p_n - p_{-n}}{\sigma})},$$

where σ is degree of horizontal product differentiation.

Bellman Equation

- $V_n(\mathbf{e})$ is the expected NPV to firm n of being in the industry given that the industry is in state $\mathbf{e} = (e_1, e_2)$.

- Bellman equation:

$$V_n(\mathbf{e}) = \max_{p_n} D_n(p_n, p_{-n}(\mathbf{e}))(p_n - c(e_n))$$
$$+ \beta \sum_{k=1}^{2} D_k(p_n, p_{-n}(\mathbf{e}))\overline{V}_{nk}(\mathbf{e}),$$

where

- $p_{-n}(\mathbf{e})$ is the price charged by the other firm;
- $\beta \in (0,1)$ is the discount factor;
- $\overline{V}_{nk}(\mathbf{e})$ is the expectation of firm n's value function conditional on buyer purchasing from firm $k \in \{1,2\}$.

- $p_n(\mathbf{e})$ is uniquely determined by FOC.

Equilibrium

- Symmetric Markov perfect equilibrium (MPE):
  - Value function $V_1^*(\mathbf{e}) = V^*(\mathbf{e})$ and $V_2^*(\mathbf{e}) = V^*(\mathbf{e}^{[2]})$ where $\mathbf{e}^{[2]}$ denotes the vector (e_2, e_1) constructed by interchanging the stocks of know-how of firms 1 and 2.
  - Policy function $p_1^*(\mathbf{e}) = p^*(\mathbf{e})$ and $p_2^*(\mathbf{e}) = p^*(\mathbf{e}^{[2]})$.
- The Bellman equation and FOC for state $\mathbf{e}$ are

$$V^*(\mathbf{e}) = D_1^*(\mathbf{e})\left(p^*(\mathbf{e}) - c(e_1)\right) + \beta \sum_{k=1}^{2} D_k^*(\mathbf{e})\overline{V}_k^*(\mathbf{e}),$$

$$0 = \sigma - (1 - D_1^*(\mathbf{e}))\left(p^*(\mathbf{e}) - c(e_1)\right) - \beta \overline{V}_1^*(\mathbf{e})$$
$$+ \beta \sum_{k=1}^{2} D_k^*(\mathbf{e})\overline{V}_k^*(\mathbf{e}).$$

This system of $2M^2$ nonlinear equations, two for each state $\mathbf{e} \in \{1, \ldots, M\}^2$, defines a symmetric equilibrium.

- Existence in pure strategies is guaranteed, uniqueness is not.

Parameterization

- We explore equilibria for the full range of progress ratios $\rho \in (0,1]$ and forgetting rates $\delta \in [0,1]$.
- Empirical estimates: $\rho \in [0.7, 0.95]$ and $\delta < 0.1$.
- Remaining parameters:

parameter	M	m	κ	σ	β
value	30	15	10	1	$\frac{1}{1.05}$

- If $\rho = 0.85$, then $c(1) = 10$, $c(2) = 8.50$, and $c(15) = \ldots = c(30) = 5.30$.
- If $\rho = 0.85$, then in the static Nash equilibrium:
  - the own-price elasticity of demand is -8.86 in state $(1, 15)$ and -2.13 in state $(15, 1)$, and
  - the cross-price elasticity of firm 1's demand with respect to firm 2's price is 2.41 in state $(15, 1)$ and 7.84 in state $(1, 15)$.
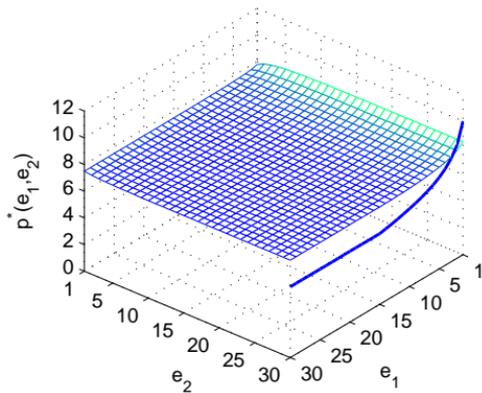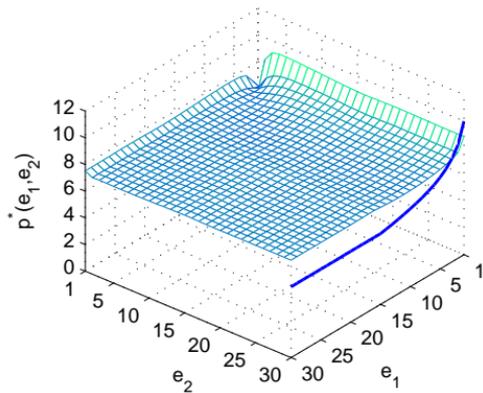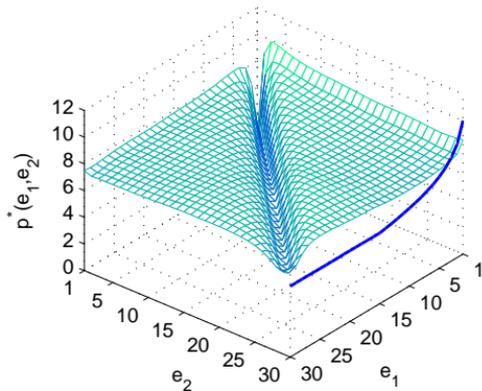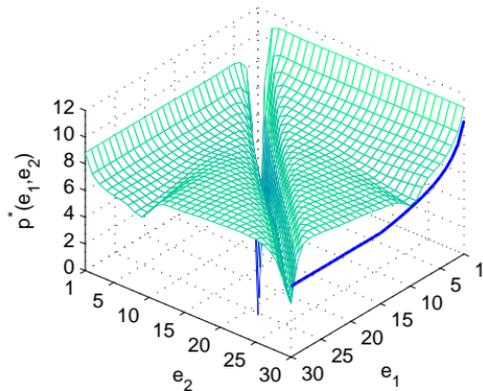
Figure 4: Policy function $p^*(e_1, e_2)$. Marginal cost $c(e_1)$ (solid line in $e_2 = 30$-plane).

Four Categories of Equilibria

- Flat equilibria:
  - Firms price near their long run marginal cost from the beginning
  - Firms do not seek to preempt each other.
- Wells:
  - Preemption battles fought by firms at the top of their learning curves.
  - Serve to build a competitive advantage $\rightarrow$ transitory advantage.
- Diagonal trenches:
  - Price wars fought by fairly symmetric firms.
  - Serve to build and defend a competitive advantage $\rightarrow$ permanent advantage.
- Sideways trenches:
  - Price wars fought by fairly asymmetric firms.
  - Serve to build and defend a competitive advantage $\rightarrow$ permanent advantage.

- *Note that representing in an understandable way the pricing policies of the firms is greatly facilitated by the industry being a duopoly.*

Industry Dynamics

- Use tools from stochastic process theory to analyze the Markov process of industry dynamics.
- Construct the probability distribution over next period's state $\mathbf{e}'$ given this period's state $\mathbf{e}$.
- Compute the distribution over states:
  - $\mu^t(\cdot)$ is the transient distribution over states in period t starting from state $(1, 1)$.
  - $\mu^\infty(\cdot)$ is the limiting (or ergodic) distribution over states.
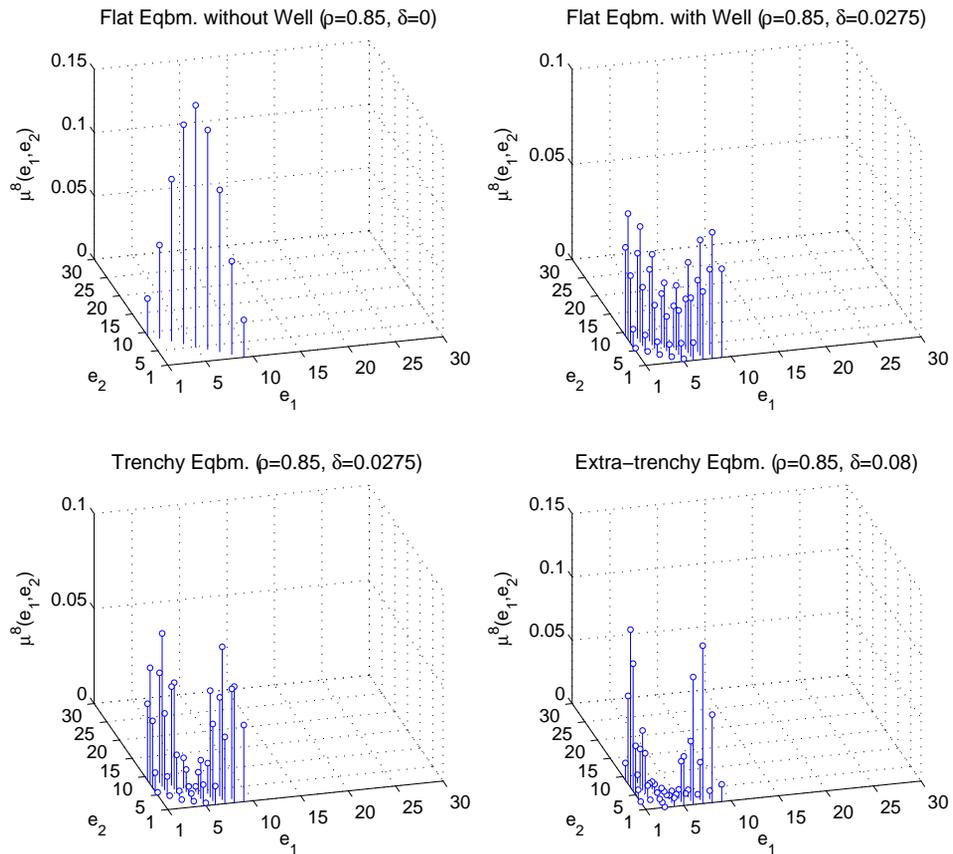
Figure 5: Transient distribution over states in period 8 given initial state $(1, 1)$.
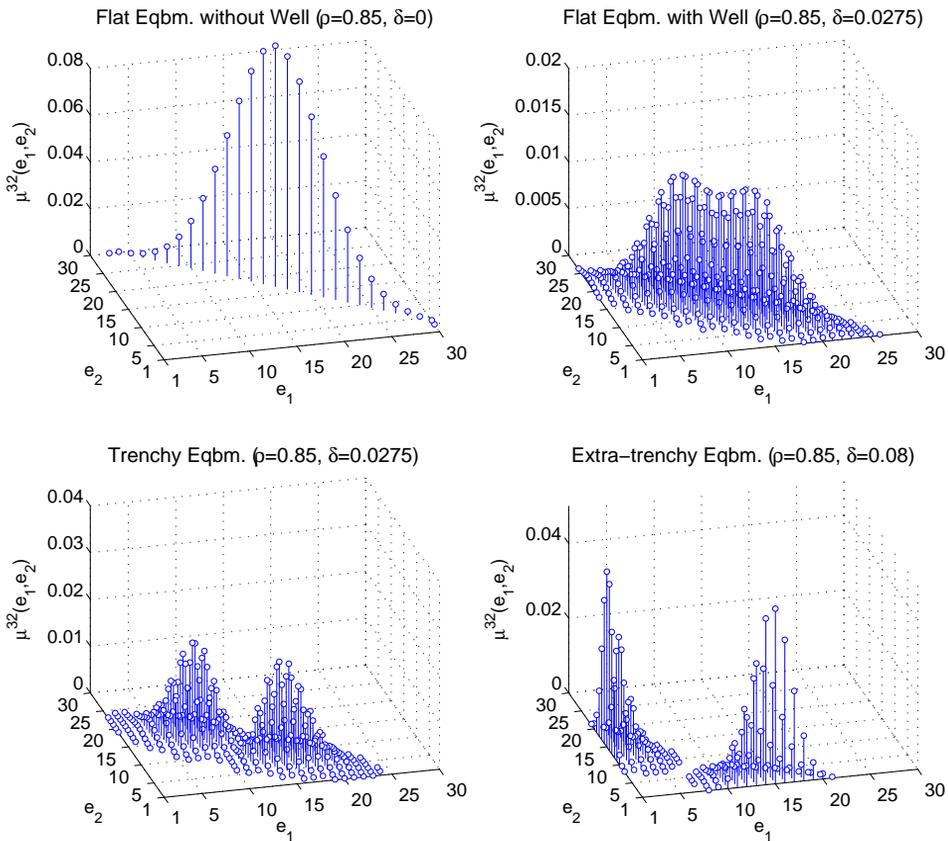
Figure 6: Transient distribution over states in period 32 given initial state $(1, 1)$.
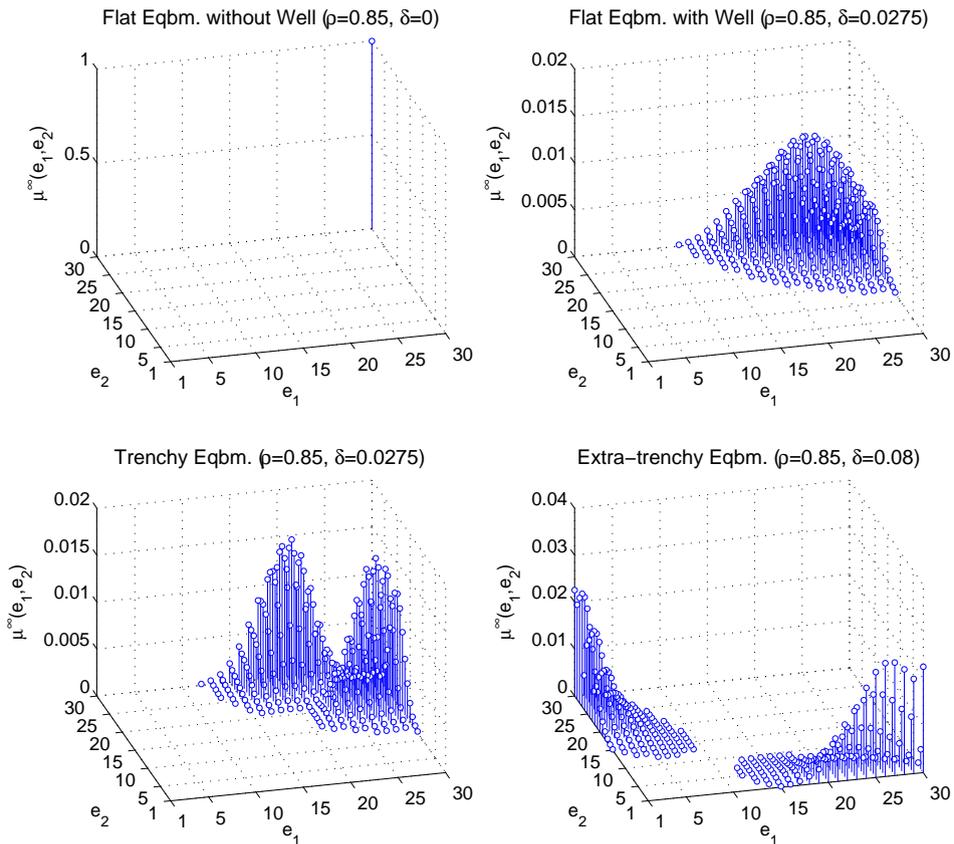
Figure 7: Limiting distribution over states.
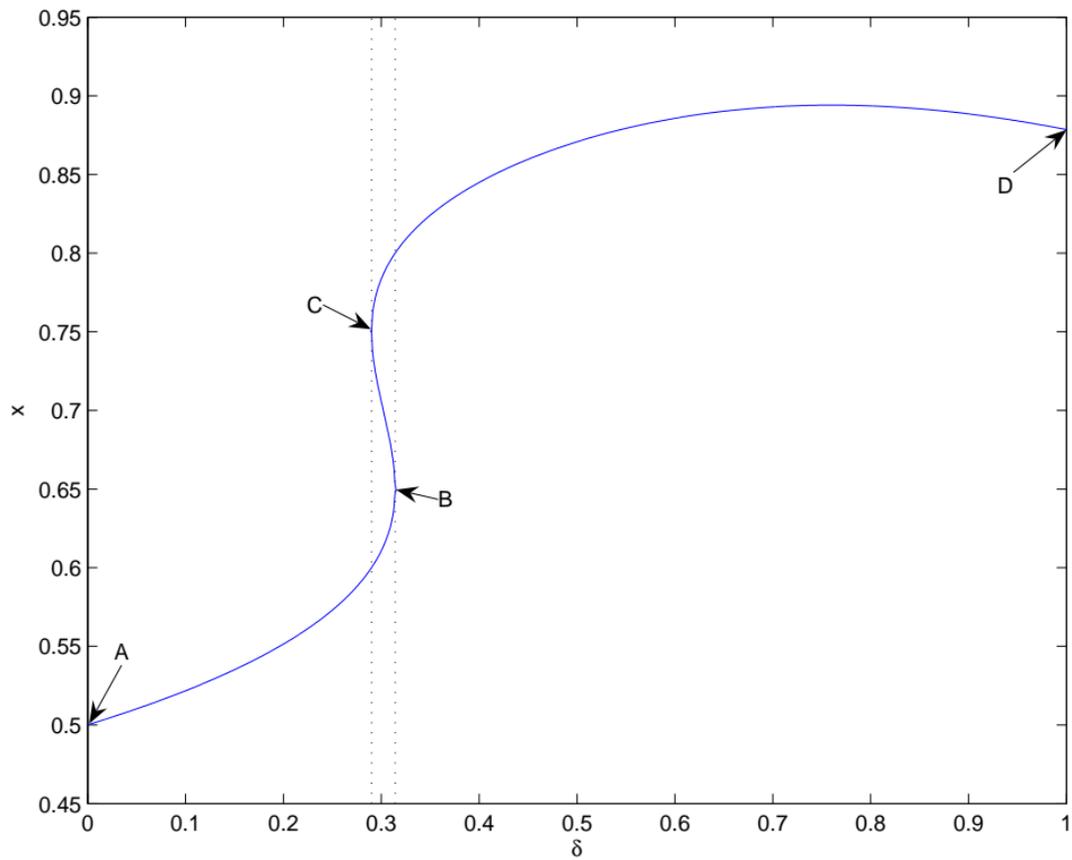
Figure 1: Homotopy example.

Homotopy Algorithm

- Write the system of $2M^2$ nonlinear equations (Bellman equations and FOCs) as

$$\mathbf{F}(\mathbf{x}, \delta) = 0,$$

where

$$\mathbf{x} = \left(V^*(1,1), V^*(2,1), \ldots, V^*(M,M), p^*(1,1), \ldots, p^*(M,M) \right).$$

- The object of interest is the equilibrium correspondence

$$\mathbf{F}^{-1} = \{ (\mathbf{x}, \delta) | \mathbf{F}(\mathbf{x}, \delta) = 0 \}.$$

- The algorithm follows a path from the unique equilibrium at $\delta = 0$ to the unique equilibrium at $\delta = 1$.

- Simple univariate example from previous slide:

$$
\begin{aligned}
f(x, \delta) &= -15.280 - \frac{\delta}{1 + \delta^4} + 67.5x - 96.923x^2 + 46.154x^3 \\
&= 0.
\end{aligned}
$$

The Example

- Define a parametric path to be a set of functions $(\mathbf{x}(s), \delta(s))$ such that $(\mathbf{x}(s), \delta(s)) \in f^{-1}$.
- The conditions that are required to remain "on path" are found by differentiating $f(x(s), \delta(s)) = 0$ with respect to s to obtain

$$\frac{\partial f(x(s), \delta(s))}{\partial x} x'(s) + \frac{\partial f(x(s), \delta(s))}{\partial \delta} \delta'(s) = 0.$$

- Solving for the ratio does not work at points C and D:

$$\frac{x'(s)}{\delta'(s)} = -\frac{\partial f(x(s), \delta(s))}{\partial \delta} \div \frac{\partial f(x(s), \delta(s))}{\partial x}.$$

- But starting at point A and solving the system of differential equations

$$x'(s) = -\frac{\partial f(x(s), \delta(s))}{\partial \delta}, \delta'(s) = \frac{\partial f(x(s), \delta(s))}{\partial x}$$

does work. Check it with $x^2 + \delta^2 = 1$ to get $x = \sin(s)$ and $y = -\cos(s)$.

The General Case

- Differentiate $\mathbf{F}(\mathbf{x}(s), \delta(s)) = 0$ with respect to s:

$$\sum_{i=1}^{2M^2} \frac{\partial F(\mathbf{x}(s), \delta(s))}{\partial x_i} x_i'(s) + \frac{\partial F(\mathbf{x}(s), \delta(s))}{\partial \delta} \delta'(s) = 0,$$

  which gives $2M^2$ differential equations in $2M^2 + 1$ unknowns $x_i'(s)$, $i = 1, \ldots, 2M^2$, and $\delta'(s)$.

- One solution is the "basic differential equations" (BDE),

$$y_i'(s) = (-1)^{i+1} \det\left(\left(\frac{\partial F(\mathbf{y}(s))}{\partial \mathbf{y}}\right)_{-i}\right),$$

$$i = 1, \ldots, 2M^2 + 1,$$

  where $\mathbf{y}(s) = (\mathbf{x}(s), \delta(s))$ and the notation $(\cdot)_{-i}$ is used to indicate that the ith column is removed from the $(2M^2 \times 2M^2 + 1)$ Jacobian $\frac{\partial F(\mathbf{y}(s))}{\partial \mathbf{y}}$. This formula is Cramer's rule.

- The BDE reduces finding new equilibria to solving an ODE with a known equilibrium as its initial condition.

Result 2 The equilibrium correspondence $\mathbf{F}^{-1}$ contains a unique path that connects the equilibrium at $\delta = 0$ with the equilibrium at $\delta = 1$. In addition, $\mathbf{F}^{-1}$ may contain (one or more) loops that are disjoint from the above path and from each other.

This result was derived by running homotopies on δ from zero to one on a 0.05 grid of ρ values. We display the results of the homotopy by plotting, for each ρ, the expected limiting Herfindahl index, H^∞, as a function of δ. Formally,

$$H^\infty = \sum_{\mathbf{e}} \left(D_1^*(\mathbf{e})^2 + D_2^*(\mathbf{e})^2 \right) \mu^\infty(\mathbf{e}).$$
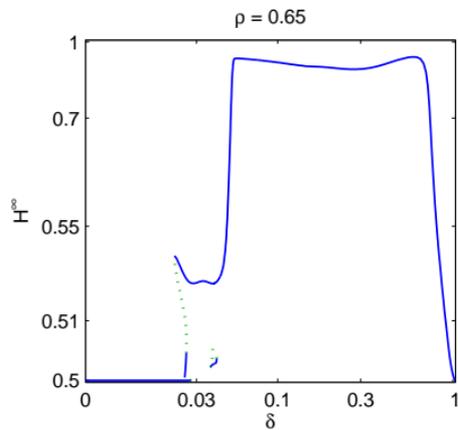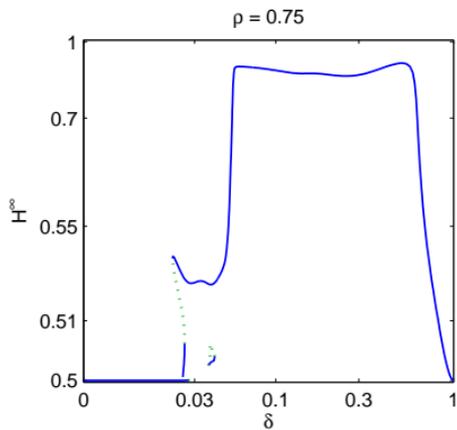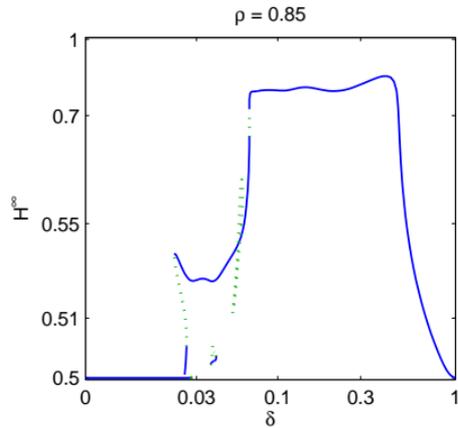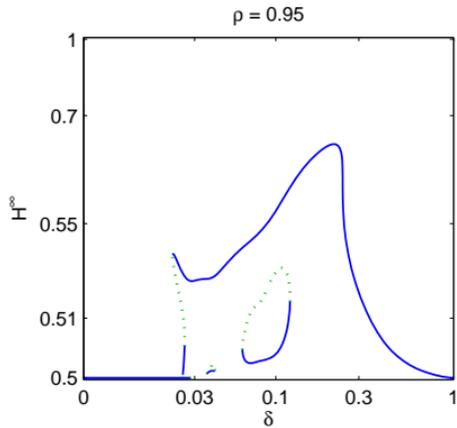
Limiting expected Herfindahl index H^∞.

Limitation of Pakes-McGuire (1994) Algorithm

- Combines value function iteration with best reply dynamics (akin to Cournot adjustment).
- Executes the iteration

$$\mathbf{x}^{l+1} = \mathbf{G}(\mathbf{x}^l), \quad l = 0, 1, 2, \ldots,$$

where, for each state $\mathbf{e} \in \{1, \ldots, M\}^2$, old guesses for the value and policy of firm 1 are mapped into new guesses.

- In between two equilibria that can be computed by the Pakes & McGuire (1994) algorithm, there is one equilibrium that cannot:

Proposition

Let $(\mathbf{x}(s), \delta(s)) \in \mathbf{F}^{-1}$. If $\delta'(s) \leq 0$, then $\rho \left(\left. \frac{\partial \mathbf{G}(\mathbf{x}(s))}{\partial \mathbf{x}} \right|_{\delta = \delta(s)} \right) \geq 1$

where here ρ is the spectral radius of the Jacobian.

Observation. Except for small systems of polynomials it is currently infeasible to prove that one has found all solutions.

Conjecture. If the equilibrium correspondence is regular and connected, then all equilibria can be identified by running homotopies along a grid of local coordinates on the manifold.
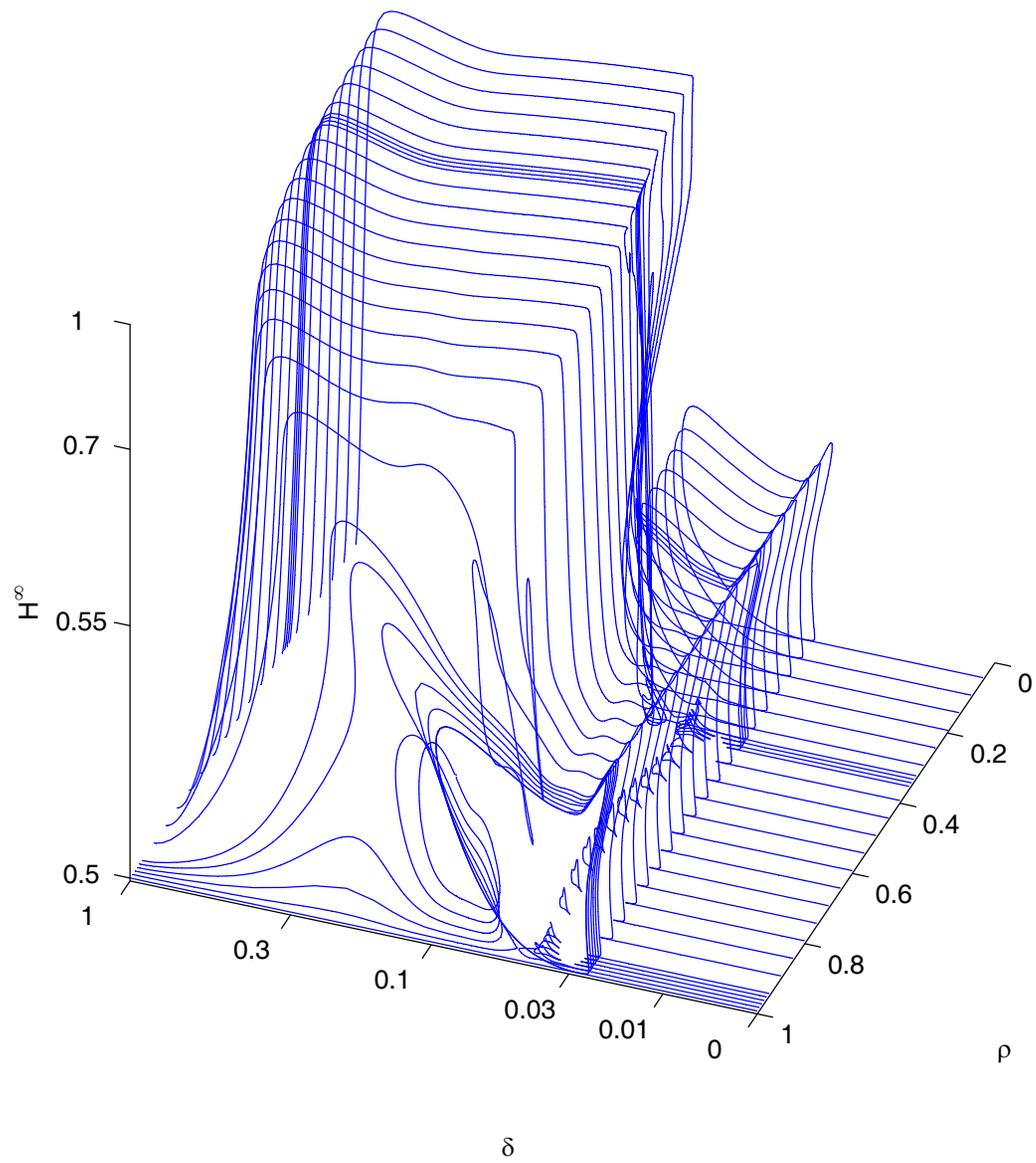
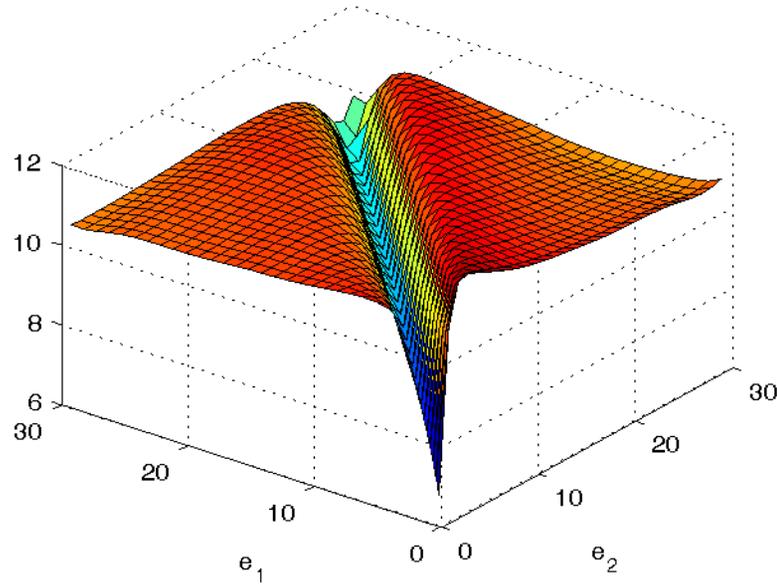Equilibrium Surface Generated by Delta Homotopies
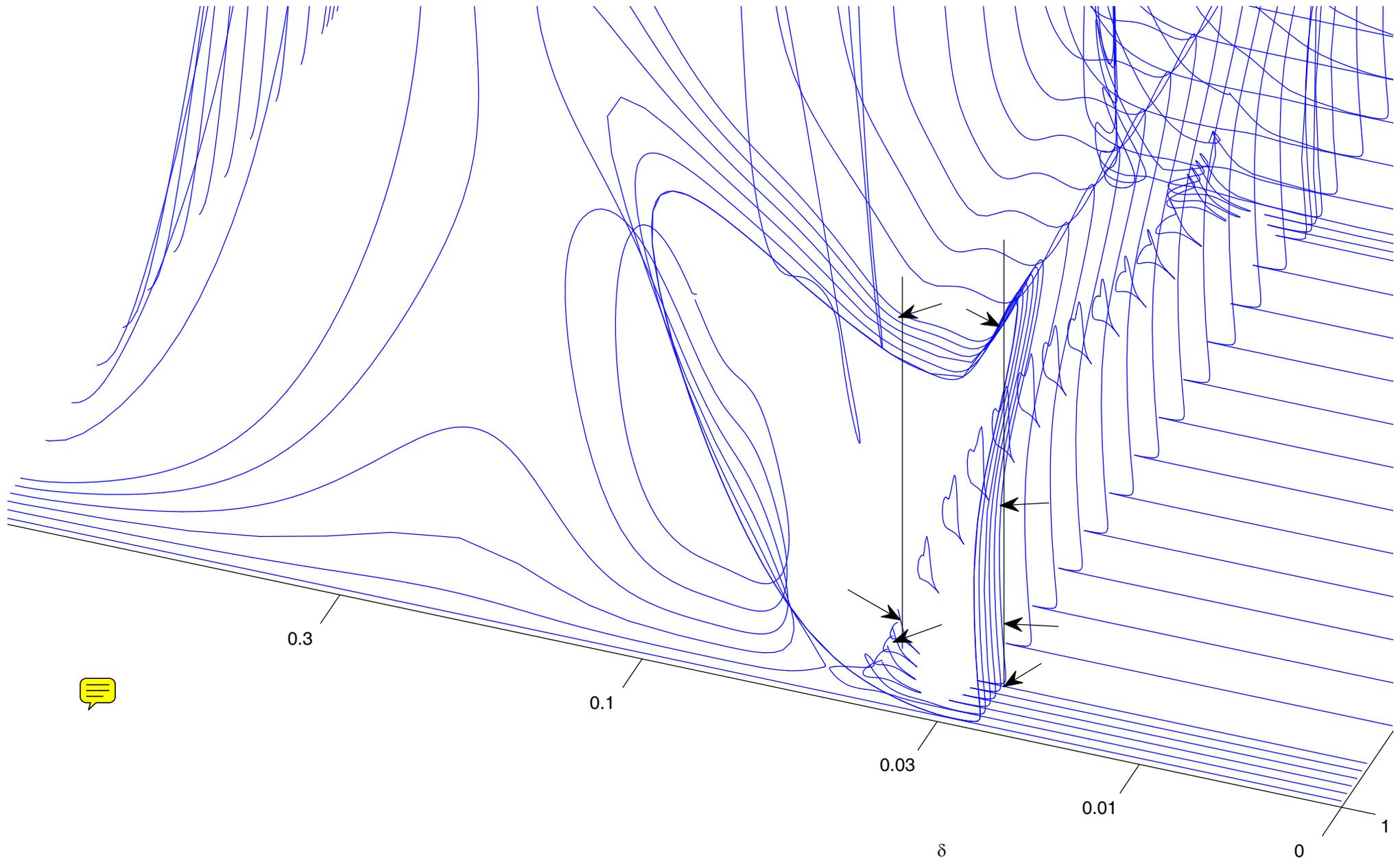
Locations of Plotted Policy Functions on Wire Representation of Equilibrium Correspondence
The vertical lines are at (0.95, 0.040) and (0.95, 0.025) in rho-delta space.

δ

0.3

0.1

0.03

0.01

1

0

Policy Functions when rho = 0.95 and delta = 0.025

Policy Functions when rho = 0.95 and delta = 0.04

Loop Above Funnel

Loop Top of Funnel

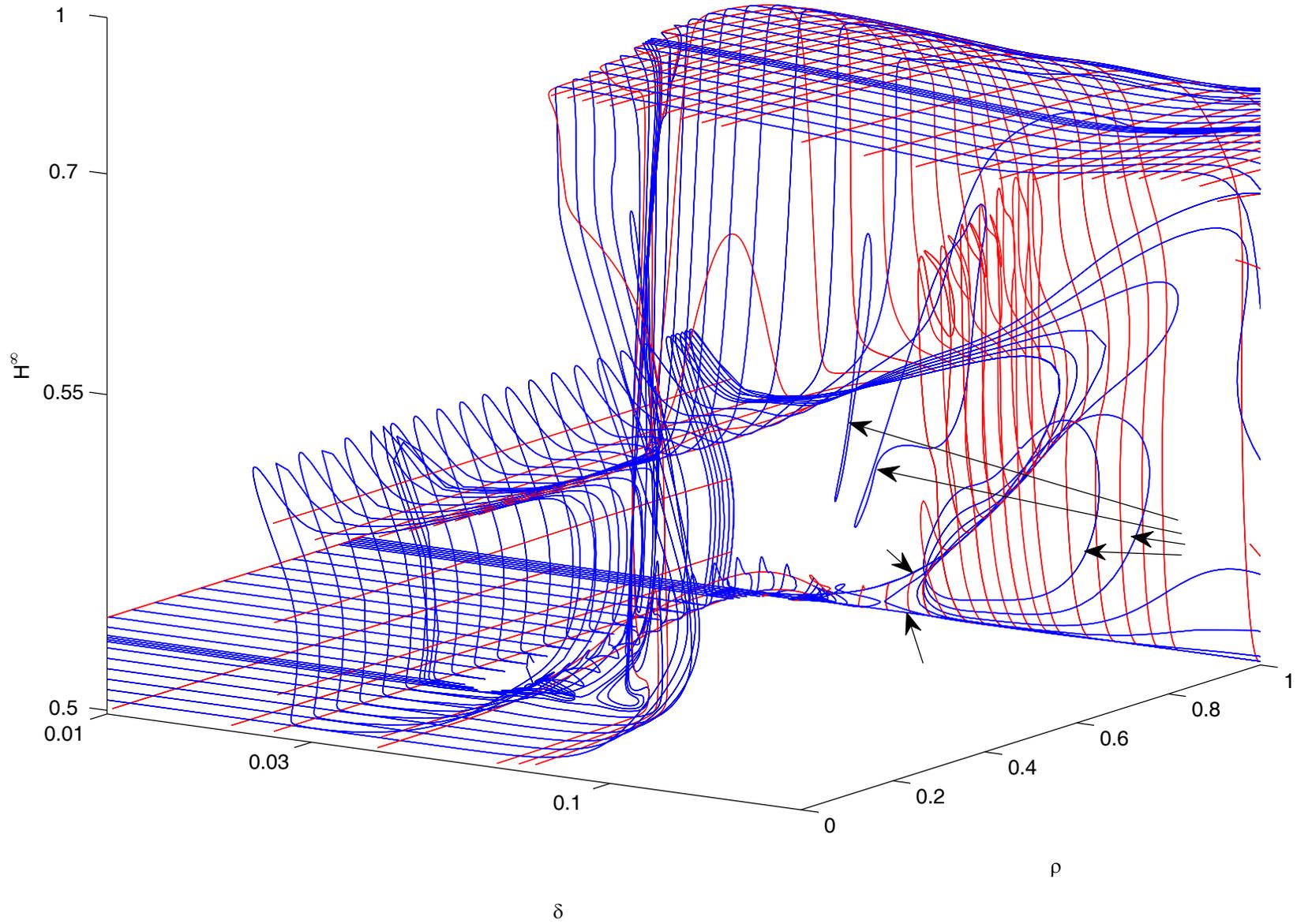Main Bot of Funnel
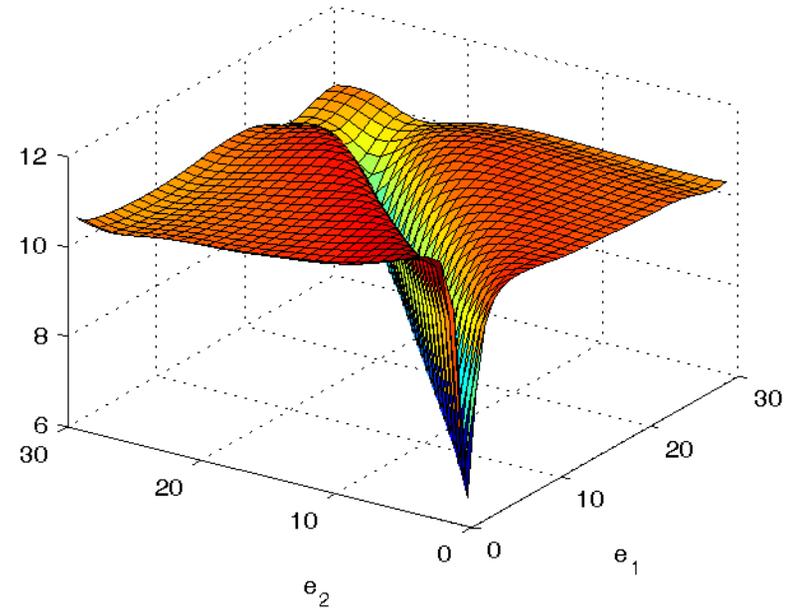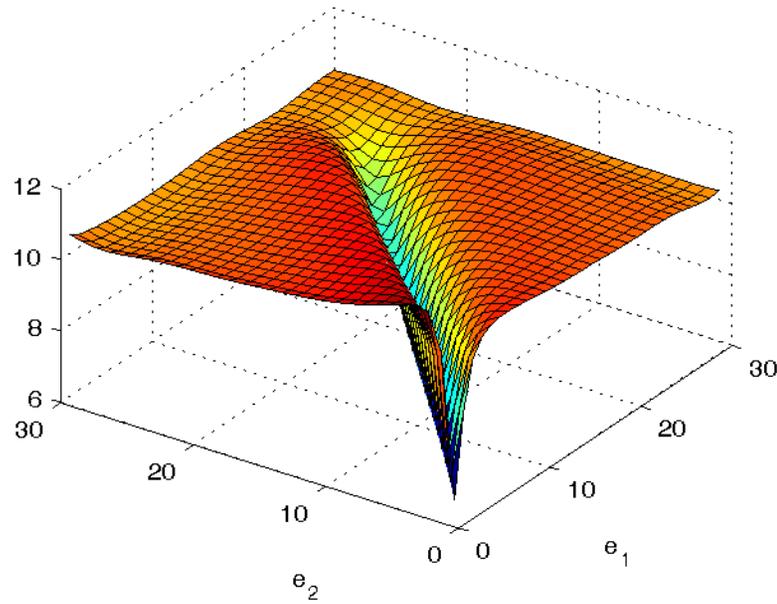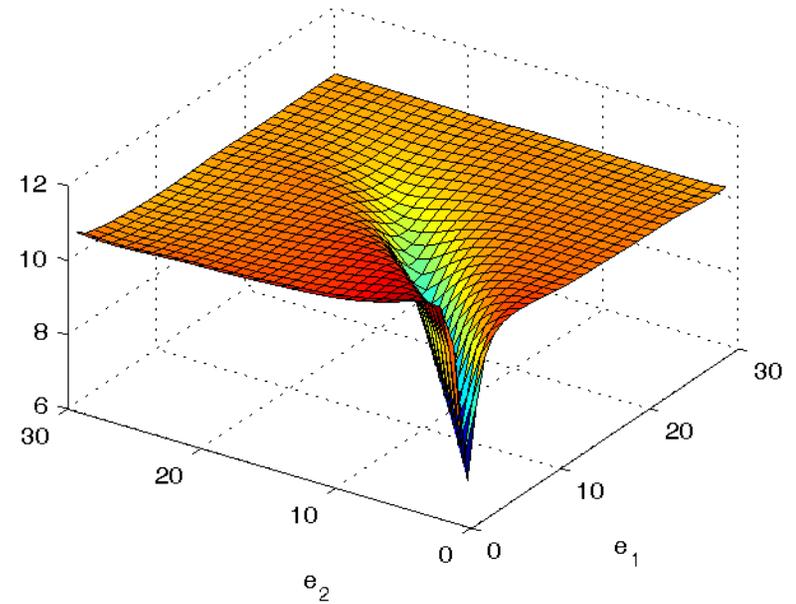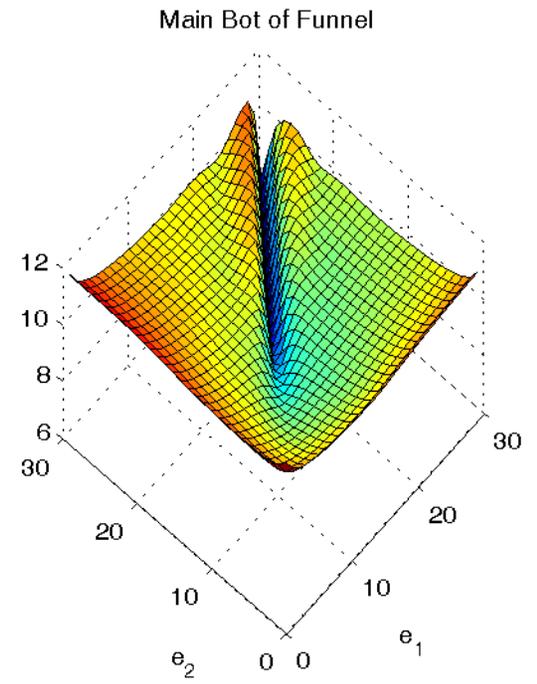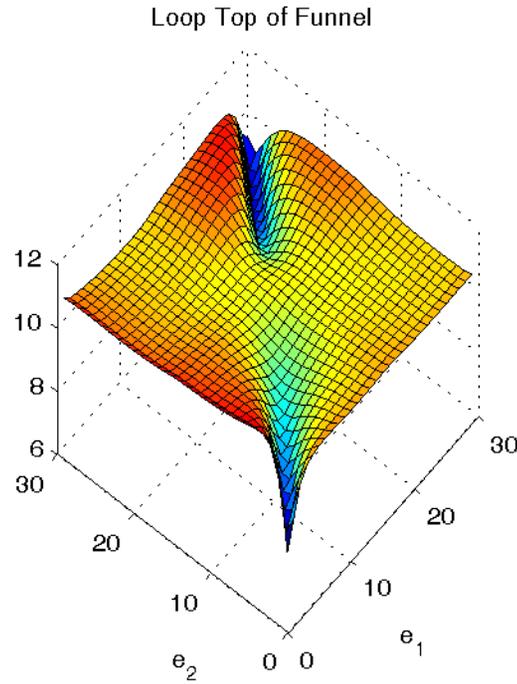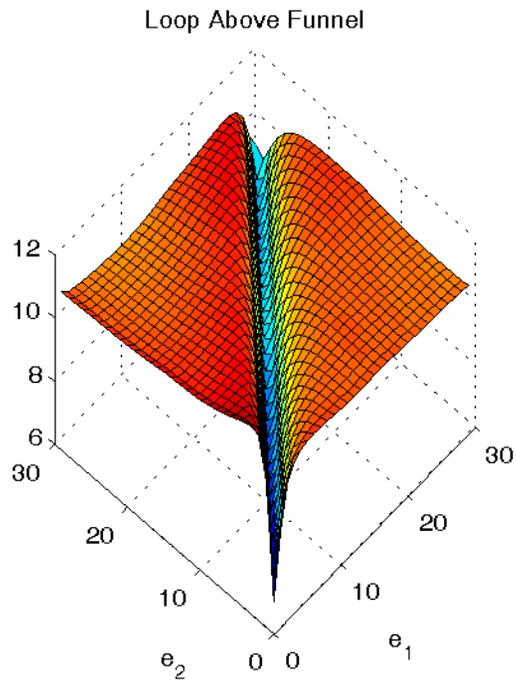
Delta [blue] and rho [red] homotopies partially trace out equilibrium correspondence.
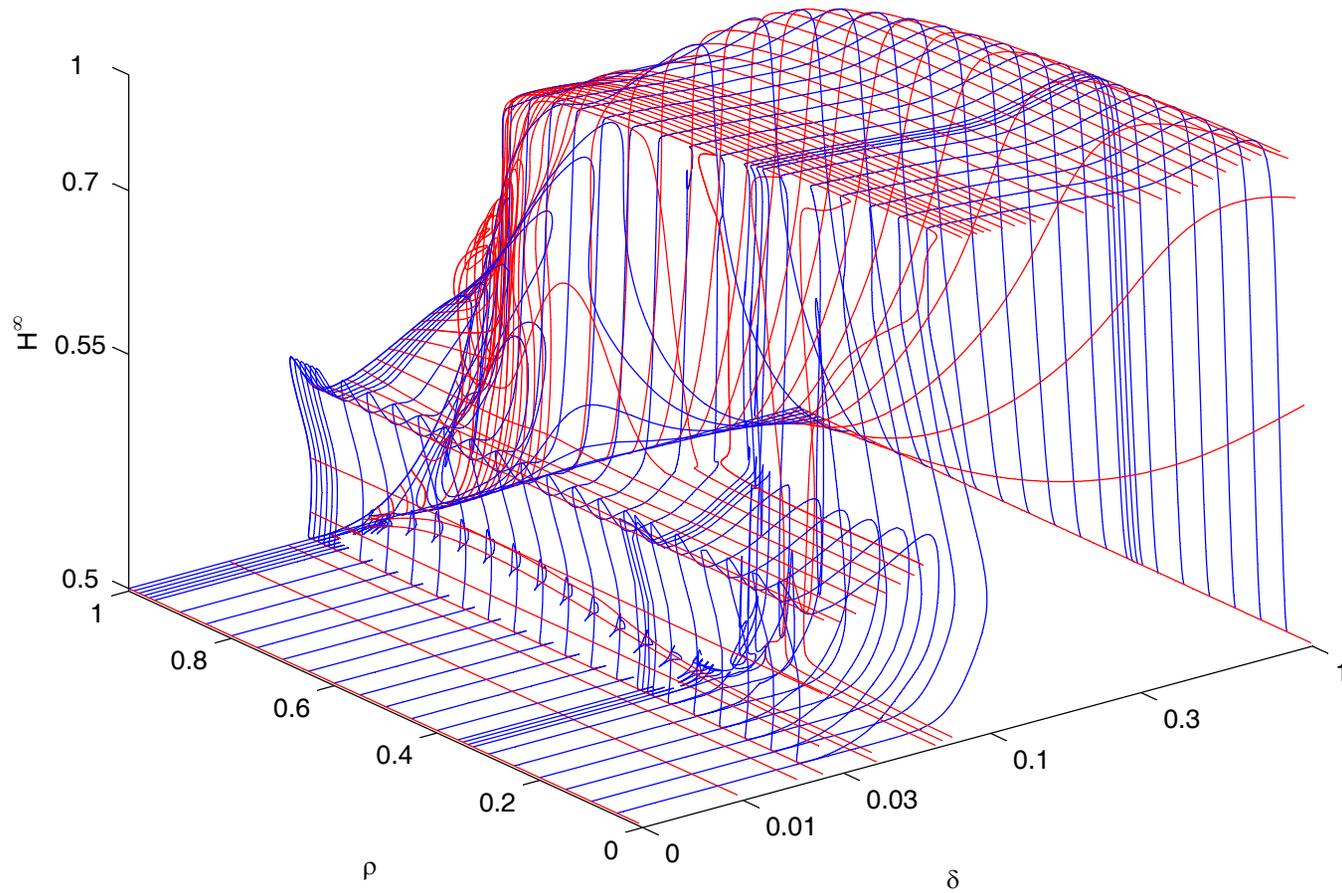
Rho homotopies trace out a rat's nest of equilibria (seen from the back).

What we have discussed?

- Goal: Discover the variety of equilibria that can occur in a duopoly in the presence of LBD and OF?
- Model
- Representation of results
- Homotopy approach to equilibrium multiplicity
  - Impossibility of Pakes-McGuire algorithm to identify more than a fraction of multiple equilibria
- Tracing out the equilibrium correspondence
  - Wealth of equilibria appear to exist.