PERTURBATION METHODS

Kenneth L. Judd

Hoover Institution and NBER

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Local Approximation Methods

- Use information about $f : R \to R$ only at a point, $x_0 \in R$, to construct an approximation valid near x_0
- Taylor Series Approximation

$$
\begin{aligned}
f(x) &\doteq f(x_0) + (x - x_0)\, f'(x_0) + \frac{(x - x_0)^2}{2}\, f''(x_0) + \cdots + \frac{(x - x_0)^n}{n!}\, f^{(n)}(x_0) + \mathcal{O}(|x - x_0|^{n+1}) \\
&= p_n\,(x) + \mathcal{O}(|x - x_0|^{n+1})
\end{aligned}
$$

- Power series: $\sum_{n=0}^{\infty} a_n z^n$
  - The *radius of convergence* is

$$
r = \sup\{|z| : |\sum_{n=0}^{\infty} a_n z^n| < \infty\},
$$

  - $\sum_{n=0}^{\infty} a_n z^n$ converges for all $|z| < r$ and diverges for all $|z| > r$.
- Complex analysis
  - $f : \Omega \subset C \to C$ on the complex plane C is *analytic* on Ω iff

$$
\forall a \in \Omega \ \ \exists r, c_k \left(\forall \, \|z - a\| < r \left(f(z) = \sum_{k=0}^{\infty} c_k (z - a)^k \right) \right)
$$

  - A *singularity of* f is any a s. t. f is analytic on $\Omega - \{a\}$ but not on Ω.
  - If f or any derivative of f has a singularity at $z \in C$, then the radius of convergence in C of $\sum_{n=0}^{\infty} \frac{(x - x_0)^n}{n!} f^{(n)}(x_0)$, is bounded above by $\| \, x_0 - z \, \|$.

- Example: $f(x) = x^\alpha$ where $0 < \alpha < 1$.
  - One singularity at $x = 0$
  - Radius of convergence for power series around $x = 1$ is 1.
  - Taylor series coefficients decline slowly:

$$a_k = \frac{1}{k!}\frac{d^k}{dx^k}(x^\alpha)|_{x=1} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{1\cdot 2\cdot\cdots\cdot k}.$$

Table 6.1 (corrected): Taylor Series Approximation Errors for $x^{1/4}$

	Taylor series error				$x^{1/4}$
x N:	5	10	20	50	
3.0	5(−1)	8(1)	3(3)	1(12)	1.3161
2.0	1(−2)	5(−3)	2(−3)	8(−4)	1.1892
1.8	4(−3)	5(−4)	2(−4)	9(−9)	1.1583
1.5	2(−4)	3(−6)	1(−9)	0(−12)	1.1067
1.2	1(−6)	2(−10)	0(−12)	0(−12)	1.0466
.80	2(−6)	3(−10)	0(−12)	0(−12)	.9457
.50	6(−4)	9(−6)	4(−9)	0(−12)	.8409
.25	1(−2)	1(−3)	4(−5)	3(−9)	.7071
.10	6(−2)	2(−2)	4(−3)	6(−5)	.5623
.05	1(−1)	5(−2)	2(−2)	2(−3)	.4729

Log-Linearization and General Nonlinear COV

- Implicit differentiation implies

$$\hat{x} = \frac{dx}{x} = -\frac{\varepsilon f_\varepsilon}{x f_x}\frac{d\varepsilon}{\varepsilon} = -\frac{\varepsilon f_\varepsilon}{x f_x}\,\varepsilon,$$

- Since $\hat{x} = d(\ln x)$, log-linearization implies log-linear approximation

$$\ln x - \ln x_0 \doteq -\frac{\varepsilon_0 f_\varepsilon(x_0,\varepsilon_0)}{x_0 f_x(x_0,\varepsilon_0)}(\ln \varepsilon - \ln \varepsilon_0). \tag{6.1.5}$$

- Generalization to nonlinear change of variables.
  - Take any monotonic $h(\cdot)$, and define $x = h(X)$ and $y = h(Y)$
  - Use the identity

$$f(Y,X) = f(h^{-1}(h(Y)), h^{-1}(h(X))) = f(h^{-1}(y), h^{-1}(x)) \equiv g(y,x).$$

    to generate expansions

$$\begin{aligned} y(x) &\doteq y(x_0) + y'(x)(x - x_0) + ... \\ Y(X) &\doteq h^{-1}\left(y(h(X_0)) + y'(h(X_0))(h(X) - h(X_0)) + ...\right) \end{aligned}$$

  - $h(z) = \ln z$ is commonly used by economists, but others may be better globally

Implicit Function Theorem

- Suppose $h : R^n \to R^m$ is defined in $H(x, h(x)) = 0$, $H : R^n \times R^m \to R^m$, and $h(x_0) = y_0$.
  - Implicit differentiation shows
$$H_x(x, h(x)) + H_y(x, h(x))h_x(x) = 0$$
  - At $x = x_0$, this implies
$$h_x(x_0) = -H_y(x_0, y_0)^{-1} H_x(x_0, y_0)$$
    if $H_y(x_0, y_0)$ is nonsingular. More simply, we express this as
$$h_x^0 = -\left(H_y^0\right)^{-1} H_x^0$$
  - Linear approximation for $h(x)$ is
$$h^L(x) \doteq h(x_0) + h_x(x_0)(x - x_0)$$
- To check on quality, we compute
$$E = \hat{H}(x, h^L(x))$$
  where $\hat{H}$ is a unit free equivalent of H. If $E < \varepsilon$, then we have an ε-solution.

- If $h^L(y)$ is not satisfactory, compute higher-order terms by repeated differentiation.
  - $D_{xx}H(x, h(x)) = 0$ implies

$$H_{xx} + 2H_{xy}h_x + H_{yy}h_xh_x + H_yh_{xx} = 0$$

  - At $x = x_0$, this implies

$$h^0_{xx} = -\left(H^0_y\right)^{-1}\left(H^0_{xx} + 2H^0_{xy}h^0_x + H^0_{yy}h^0_xh^0_x\right)$$

  - Construct the quadratic approximation

$$h^Q(x) \doteq h(x_0) + h^0_x(x - x_0) + \frac{1}{2}(x - x_0)^\top h^0_{xx}(x - x_0)$$

    and check its quality by computing $E = H(x, h^Q(x))$.

Regular Perturbation: The Basic Idea

- Suppose x is an endogenous variable, ε a parameter
  - Want to find $x(\varepsilon)$ such that $f(x(\varepsilon), \varepsilon) = 0$
  - Suppose $x(0)$ known.
- Use Implicit Function Theorem
  - Apply implicit differentiation:

$$f_x(x(\varepsilon), \varepsilon)x'(\varepsilon) + f_\varepsilon(x(\varepsilon), \varepsilon) = 0 \tag{13.1.5}$$

  - At $\varepsilon = 0$, $x(0)$ is known and (13.1.5) is linear in $x'(0)$ with solution

$$x'(0) = -f_x(x(0), 0)^{-1} f_\varepsilon(x(0), 0)$$

  - Well-defined only if $f_x \neq 0$, a condition which can be checked at $x = x(0)$.
  - The linear approximation of $x(\varepsilon)$ for ε near zero is

$$x(\varepsilon) \doteq x^L(\varepsilon) \equiv x(0) - f_x(x(0), 0)^{-1} f_\varepsilon(x(0), 0)\varepsilon \tag{13.1.6}$$

- Can continue for higher-order derivatives of $x(\varepsilon)$.
  - Differentiate (13.1.5) w.r.t. ε

$$f_x x'' + f_{xx}(x')^2 + 2f_{x\varepsilon}x' + f_{\varepsilon\varepsilon} = 0 \tag{13.1.7}$$

  - At $\varepsilon = 0$, (13.1.7) implies that

$$\begin{aligned} x''(0) = &-f_x(x(0),0)^{-1}\left(f_{xx}(x(0),0)\ (x'(0))^2\right. \\ &\left.+2f_{x\varepsilon}(x(0),0)\ x'(0) + f_{\varepsilon\varepsilon}(x(0),0)\right) \end{aligned}$$

  - Quadratic approximation is

$$x(\varepsilon) \doteq x^Q(\varepsilon) \equiv x(0) + \varepsilon x'(0) + \frac{1}{2}\varepsilon^2 x''(0) \tag{13.1.8}$$

- General Perturbation Strategy
  - Find special (likely degenerate, uninteresting) case where one knows solution
    * General relativity theory: begin with case of a universe with zero mass: ε is mass of universe
    * Quantum mechanics: begin with case where electrons do not repel each other: ε is force of repulsion
    * Business cycle analysis: begin with case where there are no shocks: ε is measure of exogenous shocks
  - Use local approximation theory to compute nearby cases
    * Standard implicit function may be applicable
    * Sometimes standard implicit function theorem will not apply; use appropriate bifurcation or singularity method.
  - Check to see if solution is good for problem of interest
    * Use unit-free formulation of problem
    * Go to higher-order terms until error is reduced to acceptable level
    * *Always* check solution for range of validity

Single-Sector, Deterministic Growth - canonical problem

- Consider dynamic programming problem

$$\max_{c(t)} \int_0^\infty e^{-\rho t} u(c) dt$$
$$\dot{k} = f(k) - c$$

- Ad-Hoc Method: Convert to a wrong LQ problem
  - McGrattan, JBES (1990)
    - Replace $u(c)$ and $f(k)$ with approximations around c^* and k^*
    - Solve linear-quadratic problem

$$\begin{array}{l} \max_c \int_0^\infty e^{-\rho t} \left(u(c^*) + u'(c^*)(c - c^*) + \frac{1}{2} u''(c^*)(c - c^*)^2 \right) \ dt \\ \text{s.t.} \quad \dot{k} = f(k^*) + f'(k^*)(k^* - k) - c \end{array}$$

    - Resulting approximate policy function is

$$C^{McG}(k) = f(k^*) + \rho(k - k^*) \neq C(k^*) + C'(k^*)(k - k^*)$$

    - Local approximate law of motion is $\dot{k} = 0$; add noise to get

$$dk = 0 \cdot dt + dz$$

    - Approximation is *random walk* when theory says solution is stationary
  - Fallacy of McGrattan noted in Judd (1986, 1988); point repeated in Benigno-Woodford (2004).

- Kydland-Prescott
  - Restate problem so that $\dot{k}$ is linear function of state and controls
  - Replace $u(c)$ with quadratic approximation
  - Note 1: such transformation may not be easy
  - Note 2: special case of Magill (JET 1977).
- Lesson
  - Kydland-Prescott, McGrattan provide no mathematical basis for method
  - Formal calculations based on appropriate IFT should be used.
  - Beware of *ad hoc* methods based on an intuitive story!

Perturbation Method for Dynamic Programming

- Formalize problem as a system of functional equations
  - Bellman equation:
$$\rho V(k) = \max_c \; u(c) + V'(k)(f(k) - c) \tag{1}$$
  - $C(k)$: policy function defined by
$$\begin{aligned} 0 &= u'(C(k)) - V'(k) \\ \rho V(k) &= u(C(k)) + V'(k)(f(k) - C(k)) \end{aligned} \tag{2}$$
  - Apply envelope theorem to (1) to get
$$\rho V'(k) = V''(k)(f(k) - C(k)) + V'(k)f'(k) \tag{1_k}$$
  - Steady-state equations
$$\begin{array}{ll} c^* = f(k^*) & \rho V(k^*) = u(c^*) + V'(k^*)(f(k^*) - c^*) \\ 0 = u'(c^*) - V'(k^*) & \rho V'(k) = V''(k)(f(k^*) - c^*) + V'(k)f'(k) \end{array}$$
  - Steady State: We know k^*, $V(k^*)$, $C(k^*)$, $f'(k^*)$, $V'(k^*)$:
$$\rho = f'(k^*), \quad C(k^*) = f(k^*), \quad V(k^*) = \rho^{-1}u(c^*), \quad V'(k^*) = u'(c^*)$$
  - Want Taylor expansion:
$$\begin{aligned} C(k) &\doteq C(k^*) + C'(k^*)(k - k^*) + C''(k^*)(k - k^*)^2/2 + ... \\ V(k) &\doteq V(k^*) + V'(k^*)(k - k^*) + V''(k^*)(k - k^*)^2/2 + ... \end{aligned}$$

- Linear approximation around a steady state
  - Differentiate $(1_k, 2)$ w.r.t. k:

$$\begin{aligned} \rho V'' &= V'''(f - C) + V''(f' - C') + V''f' + V'f'' && (1_{kk}) \\ 0 &= u''C' - V'' && (2_k) \end{aligned}$$

  - At the steady state

$$0 = -V''(k^*)C'(k^*) + V''(k^*)f'(k^*) + V'(k^*)f''(k^*) \qquad (1_k^*)$$

  - Substituting (2_k) into (1_k^*) yields

$$0 = -u''(C')^2 + u''C'f' + V'f''$$

  - Two solutions

$$C'(k^*) = \frac{\rho}{2}\left(1 \pm \sqrt{1 + \frac{4u'(C(k^*))f''(k^*)}{u''(C'(k^*))f'(k^*)f'(k^*)}}\right)$$

  - However, we know $C'(k^*) > 0$; hence, take positive solution

- Higher-Order Expansions
  - Conventional perception in macroeconomics: "perturbation methods of order higher than one are considerably more complicated than the traditional linear-quadratic case ..." – Marcet (1994, p. 111)
  - Mathematics literature: No problem (See, e.g., Bensoussan, Fleming, Souganides, etc.)
- Compute $C''(k^*)$ and $V'''(k^*)$.
  - Differentiate $(1_{kk}, 2_k)$:

$$\begin{aligned} \rho V''' &= V''''(f - C) + 2V'''(f' - C') + V''(f'' - C'') \\ &\quad + V'''f' + 2V''f'' + V'f''' \end{aligned} \tag{1_{kkk}}$$

$$0 = u'''(C')^2 + u''C'' - V''' \tag{2_{kk}}$$

  - At k^*, (1_{kkk}) reduces to

$$0 = 2V'''(f' - C') + 3V''f'' - V''C'' + V'f''' \tag{1^*_{kkk}}$$

  - Equations $(1^*_{kkk}, 2^*_{kk})$ are *LINEAR* in unknowns $C''(k^*)$ and $V'''(k^*)$:

$$\begin{pmatrix} u'' & -1 \\ V'' & -2(f' - C') \end{pmatrix} \begin{pmatrix} C'' \\ V''' \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

  - Unique solution since determinant $-2u''(f' - C') + V'' < 0$.

- Compute $C^{(n)}(k^*)$ and $V^{(n+1)}(k^*)$.

  - Linear system for order n is, for some A_1 and A_2,

$$\begin{pmatrix} u'' & -1 \\ V'' & -n(f' - C') \end{pmatrix} \begin{pmatrix} C^{(n)} \\ V^{(n+1)} \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

  - Higher-order terms are produced by solving linear systems
  - The linear system is always determinate since $-nu''(f' - C') + V'' < 0$

- Conclusion:

  - Computing first-order terms involves solving quadratic equations
  - Computing higher-order terms involves solving linear equations
  - Computing higher-order terms is easier than computing the linear term.

Accuracy Measure

Consider the one-period relative Euler equation error:

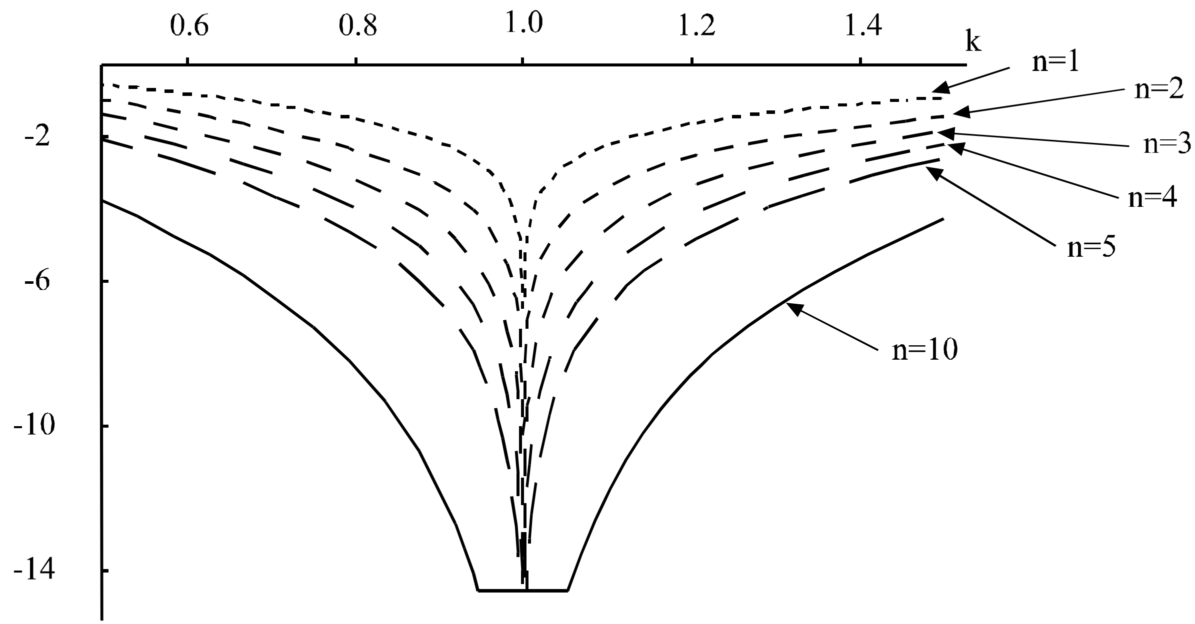
$$E(k) = 1 - \frac{V'(k)}{u'(C(k))}$$

- Equilibrium requires it to be zero.
- $E(k)$ is measure of optimization error
  - 1 is unacceptably large
  - Values such as .00001 is a limit for people.
  - $E(k)$ is unit-free.
- Define the L^p, $1 \leq p < \infty$, *bounded rationality accuracy* to be

$$\log_{10} \parallel E(k) \parallel_p$$

- The L^∞ error is the maximum value of $E(k)$.

Global Quality of Asymptotic Approximations



Graph of $\log_{10}|E(k|$

- Linear approximation is very poor even for k close to steady state
- Order 2 is better but still not acceptable for even $k = .9, 1.1$
- Order 10 is excellent for $k \in [.6, 1.4]$

Stochastic, Discrete-Time Growth

$$\begin{array}{ll} \max_{c_t} & E\left\{\sum_{t=0}^{\infty} \beta^t u(c_t)\right\} \\ \text{s.t.} & k_{t+1} = (1+\varepsilon z)F(k_t - c_t) \end{array} \tag{13.7.19}$$

- New state variable:
  - k_t is capital stock at the beginning of period t
  - consumption comes out of k
  - the remaining capital, $k_t - c_t$, is used in production
  - resulting output is $(1+\varepsilon z)F(k_t - c_t) = k_{t+1}$
  - perturbation parameter is ε, the standard deviation, not the variance.
- Do deterministic perturbation analysis.
  - Solution when $\varepsilon = 0$ is $C(k)$ solving

$$u'\left(C(k)\right) = \beta\, u'\left(C\left(F(k - C(k))\right)\right)\ F'\left(k - C(k)\right). \tag{13.7.20}$$

  - At the steady state, k^*, $F(k^* - C(k^*)) = k^*$, and $1 = \beta\, F'(k^* - C(k^*))$
  - Derivative of (13.7.20) with respect to k implies

$$\begin{aligned} u''\left(C(k)\right)\ C'(k) =& \ \beta u''\left(C\left(F(k - C(k))\right)\right)\ C'\left(F(k - C(k))\right) \\ & \times F'(k - C(k))[1 - C'(k)]\ F'(k - C(k)) \\ & + \beta\, u'\left(C\left(F(k - C(k))\right)\right)\ F''\left(k - C(k)\right)\ [1 - C'(k)] \end{aligned} \tag{13.7.21}$$

– At $k = k^*$, (13.7.21) reduces to (drop all arguments)

$$u''C' = \beta u''C'F'[1 - C']\,F' + \beta u'\,F''[1 - C']. \tag{13.7.22}$$

with stable solution

$$C' = \frac{1}{2}\left(1 - \beta - \beta^2\,\frac{u'}{u''}\,F'' + \sqrt{\left(1 - \beta - \beta^2\,\frac{u'}{u''}\,F''\right)^2 + 4\,\frac{u'}{u''}\,\beta^2\,F''}\right)$$

– Take another derivative of (13.7.21) and set $k = k^*$ to find

$$\begin{aligned} u''C'' + u'''C'C' =& \beta u'''\left(C'F'(1 - C')\right)^2\; F' + \beta u''C''\left(F'(1 - C')\right)^2\; F' \\ &+ 2\beta u''C'F'(1 - C')^2\,F'' + \beta u'F'''(1 - C')^2 \\ &+ \beta u'F''(-C''), \end{aligned}$$

which is a linear equation in the unknown $C''(k^*)$.

- Stochastic problem:
  - Euler equation is

$$u'\left(C(k)\right) = \beta\, E\left\{u'\left(g(\varepsilon,\, k,\, z)\right)\;\; R(\varepsilon,\, k, z)\right\}, \tag{13.7.23}$$

    where

$$\begin{aligned} g(\varepsilon, k, z)\;\; &\equiv C((1+\varepsilon z)F(k-C(k))), \\ R(\varepsilon,\, k,\, z) &\equiv (1+\varepsilon\, z)\, F'\left(k - C(k)\right). \end{aligned} \tag{13.7.24}$$

  - Compute C_ε
    - Differentiate (13.7.24) with respect to ε yields (we drop arguments of F and C)

$$\begin{aligned} g_\varepsilon &= C_\varepsilon + C'\left(zF - (1+\varepsilon z)F'C_\varepsilon\right), \\ g_{\varepsilon\varepsilon} &= C_{\varepsilon\varepsilon} + 2C'_\varepsilon\,\left(zF - (1+\varepsilon z)F'C_\varepsilon\right) + C''\left(zF - (1+\varepsilon z)F'C_\varepsilon\right)^2, \\ &+C'\left(-zF'C_\varepsilon\, 2 + (1+\varepsilon z)\, F''C_\varepsilon^2 - (1+\varepsilon z)\, F'C_{\varepsilon\varepsilon}\right). \end{aligned} \tag{13.7.25}$$

    - At $\varepsilon = 0$, (13.7.25) implies that

$$\begin{aligned} g_\varepsilon &= C_\varepsilon + C'(zF - F'C_\varepsilon), \\ g_{\varepsilon\varepsilon} &= C_{\varepsilon\varepsilon} + 2C'_\varepsilon(zF - F'C_\varepsilon) + C''(zF - F'C_\varepsilon)^2, \\ &+C'(-2zF'C_\varepsilon + F''C_\varepsilon^2 - F'C_{\varepsilon\varepsilon}). \end{aligned} \tag{13.7.26}$$

* Differentiate (13.7.23) with respect to ε

$$u''C_\varepsilon = \beta\, E\left\{u''g_\varepsilon(1+\varepsilon z)F' + u'F'z - u'(1+\varepsilon z)F''C_\varepsilon\right\} \tag{13.7.27}$$

$$\begin{aligned} u'''C_\varepsilon^2 + u''C_{\varepsilon\varepsilon} = \beta\, E\{ & u'''g_\varepsilon^2(1+\varepsilon z)\,F' + 2u''g_\varepsilon F'z \\ & -2u''g_\varepsilon(1+\varepsilon z)F''C_\varepsilon + u''g_{\varepsilon\varepsilon}(1+\varepsilon z)F' \\ & -2u'zF''C_\varepsilon + u'(1+\varepsilon z)F'''C_\varepsilon^2 - u'(1+\varepsilon z)F''C_{\varepsilon\varepsilon}\} \end{aligned} \tag{13.7.28}$$

* Since $E\{z\}=0$, (13.7.27) says that $C_\varepsilon = 0$, which in turn implies that

$$\begin{aligned} g_\varepsilon &= C'zF, \\ g_{\varepsilon\varepsilon} &= C_{\varepsilon\varepsilon} + 2C'_\varepsilon\, z\, F + C''(zF)^2 - C'F'C_{\varepsilon\varepsilon}. \end{aligned}$$

– Compute $C_{\varepsilon\varepsilon}$

* Second-order terms in (13.7.28), we find that at $\varepsilon = 0$,

$$\begin{aligned} u'''C_\varepsilon^2 + u''C_{\varepsilon\varepsilon} = \beta\, E\, \Big\{ & u'''g_\varepsilon^2\, F' + 2u''g_\varepsilon\, F'\, z - 2u''g_\varepsilon\, F''C_\varepsilon \\ & + u''g_{\varepsilon\varepsilon}\, F' - 2u'\, z\, F''C_\varepsilon + u'F'''C_\varepsilon^2 - u'F''C_{\varepsilon\varepsilon}\Big\} \end{aligned}$$

* Using the normalization $E\{z^2\}=1$, we find that

$$u''C_{\varepsilon\varepsilon} = \beta\left[u'''\, C'C'F^2F' + 2u''C'FF' + u''(C_{\varepsilon\varepsilon} + C''F^2 - C'F'C_{\varepsilon\varepsilon})F' - u'F''C_{\varepsilon\varepsilon}\right]$$

* Solving for $C_{\varepsilon\varepsilon}$ yields

$$C_{\varepsilon\varepsilon} = \frac{u'''\, C'C'F^2 + 2u''C'F + u''C''F^2}{u''C'F' + \beta u'F''}$$

- This exercise demonstrates that perturbation methods can also be applied to the discrete-time stochastic growth model.

Bifurcation Methods

- Suppose $H(h(\varepsilon), \varepsilon) = 0$ but $H(x, 0) = 0$ for all x.
  - IFT says
$$h'(0) = -\frac{H_\varepsilon(x_0, 0)}{H_x(x_0, 0)}$$
  - $H(x, 0) = 0$ implies $H_x(x_0, 0) = 0$, and $h'(0)$ has the form $0/0$ at $x = x_0$.
  - l'Hospital's rule implies, if which is well-defined if $H_{\varepsilon x}(x_0, 0) \neq 0$,
$$h'(0) = -\frac{H_{\varepsilon\varepsilon}(x_0, 0)}{H_{\varepsilon x}(x_0, 0)}.$$

Example: Portfolio Choices for Small Risks

- Simple asset demand model:
  - safe asset yields R per dollar invested and risky asset yields Z per dollar invested
  - If final value is $Y = W((1-\omega)R + \omega Z)$, then portfolio problem is

$$\max_{\omega} E\{u(Y)\}$$

- Small Risk Analysis
  - Parameterize cases

$$Z = R + \varepsilon z + \varepsilon^2 \pi \tag{1}$$

  - Compute $\omega(\varepsilon) \doteq \omega(0) + \varepsilon\omega'(0) + \frac{\varepsilon^2}{2}\omega''(0)$.around the deterministic case of $\varepsilon = 0$.
  - Failure of IFT: at $\varepsilon = 0$, $Z = R$, and $\omega(\ \varepsilon)$ is indeterminate, but we know that $\omega(\ \varepsilon)$ is unique for $\varepsilon \neq 0$

- Bifurcation analysis
  - The first-order condition for ω

$$0 = E\{u'\left(WR + \omega W(\varepsilon z + \varepsilon^2\pi)\right)(z + \varepsilon\pi)\} \equiv G(\omega, \varepsilon) \tag{2}$$

$$0 = G(\omega, 0), \quad \forall \omega. \tag{3}$$

  - Solve for $\omega(\varepsilon) \doteq \omega(0) + \varepsilon\omega'(0) + \frac{\varepsilon^2}{2}\omega''(0)$. Implicit differentiation implies

$$0 = G_\omega \omega' + G_\varepsilon \tag{4}$$

$$G_\varepsilon = E\{u''(Y)W(\omega z + 2\omega\varepsilon\pi)W(z + \varepsilon\pi) + u'(Y)\pi\} \tag{5}$$

$$G_\omega = E\{u''(Y)(z + \varepsilon\pi)^2\varepsilon\} \tag{6}$$

  - At $\varepsilon = 0$, $G(\omega, 0) = G_\omega(\omega, 0) = 0$ for all ω.
  - No point $(\omega, 0)$ for application of IFT to (3) to solve for $\omega'(0)$.

- We want $\omega_0 = \lim_{\varepsilon \to 0} \omega(\varepsilon)$.
  - Bifurcation theorem keys on ω_0 satisfying

$$\begin{aligned} 0 &= G_\varepsilon(\omega_0, 0) \\ &= u''(RW)\omega_0 \sigma_z^2 W + u'(RW)\pi \end{aligned} \tag{7}$$

    which implies

$$\omega_0 = -\frac{\pi}{\sigma_z^2} \frac{u'(WR)}{W u''(WR)} \tag{8}$$

  - (8) is asymptotic portfolio rule
    * same as mean-variance rule
    * ω_0 is product of risk tolerance and the risk premium per unit variance.
    * ω_0 is the limiting portfolio share as the variance vanishes.
    * ω_0 is not first-order approximation.

- To calculate $\omega'(0)$:
  - differentiate (2.4) with respect to ε

$$0 = G_{\omega\omega}\omega'\omega' + 2G_{\omega\varepsilon}\omega' + G_{\omega}\omega'' + G_{\varepsilon\varepsilon} \tag{9}$$

    where (without loss of generality, we assume $W = 1$)

$$\begin{aligned} G_{\varepsilon\varepsilon} &= E\{u'''(Y)(\omega z + 2\omega\varepsilon\pi)^2(z+\varepsilon\pi) + u''(Y)2\omega\pi(z+\varepsilon\pi) \\ &\quad +2u''(Y)(\omega z + 2\omega\varepsilon\pi)\pi\} \\ G_{\omega\omega} &= E\{u'''(Y)(z+\varepsilon\pi)^3\varepsilon\} \\ G_{\omega\varepsilon} &= E\{u'''(Y)(\omega z + 2\omega\varepsilon\pi)(z+\varepsilon\pi)^2\varepsilon + u''(Y)(z+\varepsilon\pi)2\pi\varepsilon \\ &\quad +u''(Y)(z+\varepsilon\pi)^2\} \end{aligned}$$

  - At $\varepsilon = 0$,

$$\begin{aligned} G_{\varepsilon\varepsilon} &= u'''(R)\omega_0^2 E\{z^3\} & G_{\omega\omega} &= 0 \\ G_{\omega\varepsilon} &= u''(R)E\{z^2\} \neq 0 & G_{\varepsilon\varepsilon\varepsilon} &\neq 0 \end{aligned}$$

  - Therefore,

$$\omega' = -\frac{1}{2}\frac{u'''(R)}{u''(R)}\frac{E\{z^3\}}{E\{z^2\}}\omega_0^2. \tag{10}$$

  - Equation (10) is a simple formula.
    - $\omega'(0)$ proportional to u'''/u''
    - $\omega'(0)$ proportional to ratio of skewness to variance.
    - If u is quadratic or z is symmetric, ω does not change to a first order.
  - We could continue this and compute more derivatives of $\omega(\varepsilon)$ as long as u is sufficiently differentiable.

- Other applications - see Judd and Guu (*ET*, 2001)
  - Equilibrium: add other agents, make π endogenous
  - Add assets
  - Produce a mean-variance-skewness-kurtosis-etc. theory of asset markets
  - More intuitive approach to market incompleteness then counting states and assets