Solving Dynamic Games with Newton's Method

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2006 Institute on Computational Economics
Argonne National Laboratory
July 21, 2006

Motivation

Stochastic, finite-state dynamic games have many applications in economics

Arise frequently in imperfect competition models

- Merger analysis (Gowrisankaran, 1999)
- Learning by doing (Benkard, 2000)
- Collusion (Fershtman and Pakes, 2000, de Roos, 2004)
- Capacity games (Besanko and Doraszelski, 2004)
- Advertising (Doraszelski and Markovich, 2005)

Solving Interesting Models

Numerical methods needed for solving non-trivial models

- Pakes and McGuire (1994, 2001)
- Doraszelski and Judd (2005)

Problem: Computational costs restrict applications

Our paper: We propose a simple method for solving large models

Overview of this Talk

- Description of general discrete-time stochastic games
- Basic idea of existing methods
- Newton method
- Application: Two-firm example with investment and production
- Conclusion: It is feasible to solve large games

Discrete-Time Dynamic Game

Stochastic discrete-time dynamic game (for two players)

State variables

- Represent production capacity, efficiency, experience, etc.
- State of firm i at time t is ω_t^i
- State of game is $\omega_t = \left(\omega_t^1, \omega_t^2\right) \in \Omega$

Actions

- Represent output, price decision, investments, etc.
- Firm i's action at time t is $x_t^i \geq 0$
- Collection of actions at t is $x_t = \left(x_t^1, x_t^2\right)$

Discrete-Time Dynamic Game II

Stochastic process of state-to-state transitions

- Represents uncertainty about investment success, depreciation, etc.
- Transition probabilities

$$\Pr\left(\omega_{t+1} = \xi | \omega_t, x\right) = \Pr^1\left(\omega_{t+1}^1 = \xi^1 | \omega_t^1, x_t^1\right) \cdot \Pr^2\left(\omega_{t+1}^2 = \xi^2 | \omega_t^2, x_t^2\right).$$

- Independent transitions, each firm controls its state

Payoffs

- Represent net profits from current sales, investment expenditures, etc.
- Firm i receives $\pi^i(x_t, \omega_t)$ at time t

Discrete-Time Dynamic Game III

Objective functions

– Represent total profit over an infinite horizon

$$E\left\{\sum_{t=0}^{\infty}\beta^t\pi^i\left(x_t,\omega_t\right)\right\}$$

– Both firms simultaneously maximize respective total profits

Pure Markov Strategies

Firm i uses a strategy of feedback form, $X^i(\omega)$

Firm i's expected net present value $V^i(\omega)$

Bellman equations for the two firms

$$V^1(\omega) = \max_{x^1} \pi^1\left(x^1, X^2(\omega), \omega\right) + \beta E\left\{V^1(\omega') \,|\omega, x^1, X^2(\omega)\right\}$$

$$V^2(\omega) = \max_{x^2} \pi^2\left(x^2, X^1(\omega), \omega\right) + \beta E\left\{V^2(\omega') \,|\omega, x^2, X^1(\omega)\right\}$$

Firm i's strategy, $X^i(\omega)$, is arg max of Bellman equation

$$X^1(\omega) = \arg\max_{x^1} \pi^1\left(x^1, X^2(\omega), \omega\right) + \beta E\left\{V^1(\omega') \,|\omega, x^1, X^2(\omega)\right\}$$

$$X^2(\omega) = \arg\max_{x^2} \pi^2\left(x^2, X^1(\omega), \omega\right) + \beta E\left\{V^2(\omega') \,|\omega, x^2, X^1(\omega)\right\}$$

Markov-Perfect Equilibrium

Markov-perfect ("feedback") equilibrium $(V^1(\omega), x^1(\omega), V^2(\omega), x^2(\omega))$

is a solution to the collection of Bellman equations and strategy equations

Existence: Few applications have existence theorem for pure strategy equilibria

Doraszelski and Satterthwaite (2003)

Multiplicity: A common problem; here we aim to find just one

Judd and Schmedders (2005)

Standard Gauss-Seidel Method

Initialize: Order states $\omega \in \Omega$ and make initial guesses $V^i(\omega)$ and $X^i(\omega)$

Iterate: Make many passes through Ω, updating values and strategies

$$
\begin{aligned}
X^1(\omega) &\leftarrow \arg\max_{x^1} & \pi^1\left(x^1, X^2(\omega), \omega\right) + \beta E\left\{V^1(\omega') \,|\omega, x^1, X^2(\omega)\right\} \\
V^1(\omega) &\leftarrow \max_{x^1} & \pi^1\left(x^1, X^2(\omega), \omega\right) + \beta E\left\{V^1(\omega') \,|\omega, x^1, X^2(\omega)\right\} \\
X^2(\omega) &\leftarrow \arg\max_{x^2} & \pi^2\left(x^2, X^1(\omega), \omega\right) + \beta E\left\{V^2(\omega') \,|\omega, x^2, X^1(\omega)\right\} \\
V^2(\omega) &\leftarrow \max_{x^2} & \pi^2\left(x^2, X^1(\omega), \omega\right) + \beta E\left\{V^2(\omega') \,|\omega, x^2, X^1(\omega)\right\}
\end{aligned}
$$

Basically a best-reply approach

Better than Gauss-Jacobi – a.k.a. value function iteration – which does not update $V^i(\omega)$ and $X^i(\omega)$ until next iterates are computed at all ω

Newton Method for Discrete-Time Game

Construct system of equations

– One equation for each value function in each state ω

$$V^1(\omega) = \pi^1\left(X^1(\omega), X^2(\omega), \omega\right) + \beta E\left\{V^1(\omega') \,|\omega, X^1(\omega), X^2(\omega)\right\}$$
$$V^2(\omega) = \pi^2\left(X^1(\omega), X^2(\omega), \omega\right) + \beta E\left\{V^2(\omega') \,|\omega, X^1(\omega), X^2(\omega)\right\}$$

– First-order conditions of optimality of firms' decisions in each state ω

$$\frac{\partial}{\partial x^1}\left(\pi^1\left(X^1(\omega), X^2(\omega), \omega\right) + \beta E\left\{V^1(\omega') \,|\omega, X^1(\omega), X^2(\omega)\right\}\right) \le 0$$
$$X^1(\omega) \ge 0$$
$$\frac{\partial}{\partial x^2}\left(\pi^2\left(X^1(\omega), X^2(\omega), \omega\right) + \beta E\left\{V^2(\omega') \,|\omega, X^1(\omega), X^2(\omega)\right\}\right) \le 0$$
$$X^2(\omega) \ge 0$$

Technical Issues

Large system of nonlinear equations and inequalities

Presence of complementarity conditions

Size of the Jacobian

Two Firms: Cournot Competition

Two firms produce the same good

In each period firms play a Cournot game and produce quantities q_1, q_2

Total quantity $q = q_1 + q_2$

Inverse demand function $P(q) = A - \phi q$

Firms' cost functions $C_i(c_i, q_i) = c_i\, q_i^2$

Technology of firm i given by c_i

Profits π_i for firm i

$$\pi_1(q_1, q_2; c_1) = q_1 P(q_1 + q_2) - c_1 q_1^2$$
$$\pi_2(q_1, q_2; c_2) = q_2 P(q_1 + q_2) - c_2 q_2^2$$

Static Nash Equilibrium

Static Nash equilibrium can be solved in closed-form

$$q_1^N(c_1, c_2) = A\frac{2c_2 + \phi}{4c_1c_2 + 4(c_1 + c_2)\phi + 3\phi^2}$$

$$q_2^N(c_1, c_2) = A\frac{2c_1 + \phi}{4c_1c_2 + 4(c_1 + c_2)\phi + 3\phi^2}$$

Cournot equilibrium profits

$$\pi_1^N(c_1, c_2) = \frac{A^2(c_1 + \phi)(2c_2 + \phi)^2}{(4c_1c_2 + 4(c_1 + c_2)\phi + 3\phi^2)^2}$$

$$\pi_2^N(c_1, c_2) = \frac{A^2(c_2 + \phi)(2c_1 + \phi)^2}{(4c_1c_2 + 4(c_1 + c_2)\phi + 3\phi^2)^2}$$

Dynamic Model

Firm i can affect production cost c_i through investment

For simplicity: $c_i = \frac{1}{M_i}$ where M_i is the number of machines of firm i

M_i depends on investment effort and depreciation

Increase in M_i through investment, decrease in M_i through depreciation

Probability of depreciation shock δ

Cost of investment effort u_i is $C_i(u_i) = \gamma_i u_i + \eta_i (u_i)^2$

Observe $C_i'(0) = \gamma_i$

Distinguish production cost $c_i = \frac{1}{M_i}$ and investment cost $C_i(u_i)$

Stochastic Transition Process

Number of machines $M_i \in \{1, 2, \ldots, N\}$

Popular specification of transition probabilities for $2 \leq M_i \leq N - 1$

$$\Pr^i(M_i^+ | M_i, u_i) = \begin{cases} \frac{(1-\delta)\alpha u_i}{1+\alpha u_i} & \xi^i = M_i + 1 \\ \frac{1-\delta+\delta\alpha u_i}{1+\alpha u_i} & \xi^i = M_i \\ \frac{\delta}{1+\alpha u_i} & \xi^i = M_i - 1 \end{cases}$$

State-to-state transition probabilities

$$\Pr\big((M_1^+, M_2^+) | (M_1, M_2), (u_1, u_2)\big) = \Pr^1(M_1^+ | M_1, u_1) \cdot \Pr^2(M_2^+ | M_2, u_2)$$

Complete Dynamic Game

State of the economy is (M_1, M_2) at the beginning of period

Production technologies of firms $(c_1, c_2) = (\frac{1}{M_1}, \frac{1}{M_2})$

Cournot outcome on product market with period profits (π_1^N, π_2^N)

Firms' investment in technology (u_1, u_2) incurring costs $(C_1(u_1), C_2(u_2))$

Stochastic transition to new states (M_1^+, M_2^+) for next period

Infinite-horizon model

Firms have discount factor β

Firms maximize expected discounted sum of per-period profits

Optimality Conditions

Separation between static Cournot game and dynamic investment decisions

Optimal investment effort $U_1(M_1, M_2)$ satisfies

$$V_1(M_1, M_2) = \left(\pi_1^N(M_1, M_2) - C_1(U_1(M_1, M_2))\right)$$
$$+ \beta \sum_{M_1^+} \sum_{M_2^+} \Pr^1(M_1^+|M_1, U_1(M_1, M_2)) \cdot \Pr^2(M_2^+|M_2, U_2(M_1, M_2)) V_1(M_1^+, M_2^+)$$

If $U_1(M_1, M_2) > 0$ then

$$0 = -\frac{\partial}{\partial u_1} C_1(U_1(M_1, M_2))$$
$$+ \beta \sum_{M_1^+} \sum_{M_2^+} \frac{\partial}{\partial u_1} \Pr^1(M_1^+|M_1, U_1(M_1, M_2)) \cdot \Pr^2(M_2^+|M_2, U_2(M_1, M_2)) V_1(M_1^+, M_2^+)$$

If $U_1(M_1, M_2) = 0$ then

$$0 \geq -\frac{\partial}{\partial u_1} C_1(U_1(M_1, M_2))$$
$$+ \beta \sum_{M_1^+} \sum_{M_2^+} \frac{\partial}{\partial u_1} \Pr^1(M_1^+|M_1, U_1(M_1, M_2)) \cdot \Pr^2(M_2^+|M_2, U_2(M_1, M_2)) V_1(M_1^+, M_2^+)$$

Solutions

Recall cost of investment effort $C_1(u_1) = \gamma_1 u_1 + \eta_1 (u_1)^2$

If $\gamma_1 = 0$ then interior solution $u_1 > 0$ and no complementarity conditions necessary

If $\gamma_1 > 0$ then boundary solution $u_1 = 0$ possible

Four equations for each state (M_1, M_2), so $4 \times N^2$ equations

Running times in seconds (using the PATH solver)

γ_1	γ_2	$N = 20$	$N = 50$	$N = 80$	$N = 100$
0	0	0.56	11.2	72	146
1	1	0.57	12.5	59	192
1	2	0.62	12.8	98	182

More Interesting Models

Cournot stage game was solved in closed-form

No analytical solution for Cournot quantity q_i for more general functions

Replace

$$V_1(M_1, M_2) = \arg\max_{u_1} \left(\pi_1^N(M_1, M_2) - C_1(u_1)\right)$$
$$+ \beta \sum_{M_1^+} \sum_{M_2^+} \Pr{}^1(M_1^+|M_1, u_1) \cdot \Pr{}^2(M_2^+|M_2, U_2(M_1, M_2))V_1(M_1^+, M_2^+)$$

by

$$V_1(M_1, M_2) = \arg\max_{u_1, q_1} (\pi_1(q_1, Q_2(M_1, M_2); M_1) - C_1(u_1))$$
$$+ \beta \sum_{M_1^+} \sum_{M_2^+} \Pr{}^1(M_1^+|M_1, u_1) \cdot \Pr{}^2(M_2^+|M_2, U_2(M_1, M_2))V_1(M_1^+, M_2^+)$$

where

$$\pi_1(q_1, q_2; M_1) = q_1 P(q_1 + q_2) - \frac{1}{M_1} q_1^2$$

More Equations

Additional optimality conditions

If $Q_1(M_1, M_2) > 0$ then

$$\frac{\partial}{\partial q_1}\pi_1(Q_1(M_1, M_2), Q_2(M_1, M_2); M_1) = 0$$

If $Q_1(M_1, M_2) = 0$ then

$$\frac{\partial}{\partial q_1}\pi_1(Q_1(M_1, M_2), Q_2(M_1, M_2); M_1) \leq 0$$

Additional complementarity conditions

Solving More Equations

Six equations for each state (M_1, M_2), so $6 \times N^2$ equations

Production quantities are always positive (complementarity conditions not needed)

Running times in seconds

γ_1	γ_2	$N = 20$	$N = 50$	$N = 80$	$N = 100$
0	0	0.65	13.9	62	128
1	1	0.65	14.4	112	287
1	2	0.70	15.3	86	234

No significant difference to smaller systems with explicit profit functions

Summary

Stochastic dynamic discrete-time games with thousands of states

Explicit solution for the static Nash equilibrium unnecessary

Multi-dimensional controls

Complementarity conditions

Corner solutions

Next Steps

More general cost functions

Multi-dimensional state vectors per player

More players

More general transitions (jump more than one unit per state)

Specialized version of PATH: better scaling and linear algebra routines