Numerical Optimization for Economists

Todd Munson
Mathematics and Computer Division
Argonne National Laboratory
tmunson@mcs.anl.gov

Institute for Computational Economics
University of Chicago
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Part I

Numerical Optimization: Introduction

Modeling Language Benefits

- Portable language for optimization problems
  - Algebraic description
  - Models easily modified and solved
  - Large problems can be processed
  - Programming language features

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  - Relational databases and spreadsheets
  - MATLAB interface for function evaluations

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  - Algorithms specific options can be set
- Communication with other tools
  - Relational databases and spreadsheets
  - MATLAB interface for function evaluations
- Excellent documentation
- Large user communities

Modeling Languages Versus Solver Libraries

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    - Select appropriate matrix representation
    - Reorder data for locality of reference
  - Memory requirements are less

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- Solver libraries
  - Computationally efficient
    - Select appropriate matrix representation
    - Reorder data for locality of reference
  - Memory requirements are less
  - Very time consuming to code and validate application
  - Limited to a single numerical method

Model Declaration

- Sets
  - Unordered, ordered, and circular sets
  - Cross products and point to set mappings
  - Set manipulation
- Parameters and variables
  - Initial and default values
  - Lower and upper bounds
  - Check statements
  - Defined variables
- Objective function and constraints
  - Equality, inequality, and range constraints
  - Complementarity constraints
  - Multiple objectives
- Problem statement

Data and Commands

- Data declaration
  - Set definitions
    - Explicit list of elements
    - Implicit list in parameter statements
  - Parameter definitions
    - Tables and transposed tables
    - Higher dimensional parameters
- Execution commands
  - Load model and data
  - Select problem, algorithm, and options
  - Solve the instance
  - Output results
- Other operations
  - Let and fix statements
  - Conditionals and loop constructs
  - Execution of external programs

Part II

Numerical Optimization I: Static Models

Model Formulation

- Classify m people into two groups using v variables
  - $c \in \{0,1\}^m$ is the known classification
  - $d \in \Re^{m \times v}$ are the observations
  - $\beta \in \Re^{v+1}$ defines the separator
  - logit distribution function
- Maximum likelihood problem

$$\max_{\beta} \quad \sum_{i=1}^{m} c_i \log(f(\beta, d_{i,\cdot})) + (1 - c_i) \log(1 - f(\beta, d_{i,\cdot}))$$

where

$$f(\beta, x) = \frac{\exp\left(\beta_0 + \sum_{j=1}^{v} \beta_j x_j\right)}{1 + \exp\left(\beta_0 + \sum_{j=1}^{v} \beta_j x_j\right)}$$

Model: mle.mod

```
param m > 0, integer;            # Population
param v > 0, integer;            # Variables

param c {1..m} binary;           # Classification
param d {1..m, 1..v};            # Observation data

var beta {0..v};                 # Weights

var lcomb {i in 1..m} = beta[0] + sum{j in 1..v} beta[j]*D[i,j];
var logit {i in 1..m} = exp(lcomb[i]) / (1+exp(lcomb[i]));

maximize likelihood:
  sum {i in 1..m} (c[i]*log(logit[i]) + (1-c[i])*log(1-logit[i]));
```

Data: mle.dat

```
param m := 3000;          # Population
param v := 3;             # Variables -- age, income, gender

# Random classification and observations
let {i in 1..m} c[i] := floor(Uniform(0,1) + 0.5);
let {i in 1..m, j in 1..v-1} d[i,j] := Uniform(0,1);
let {i in 1..m} d[i,v] := floor(Uniform(0,1) + 0.5);
```

Commands: mle.cmd

```
# Load model and data
model mle.mod;
data mle.dat;

# Specify solver and options
option solver tron;
option tron_options "frtol=0 fatol=0 gtol=1e-8";

# Solve the instance
solve;

# Output the results
display beta;
```

Output

```
ampl: include mle.cmd;
TRON: frtol=0
fatol=0
gtol=1e-8

Projected gradient at final iterate                5.95e-08
Function value at final iterate                   2077.6373

Total execution time                               0.03 sec

Exit Message     CONVERGENCE: GTOL TEST SATISFIED

beta [*] :=
0  -0.186925
1   0.161719
2   0.15411
3   0.0509116
;
ampl: quit;
```

Solution Techniques

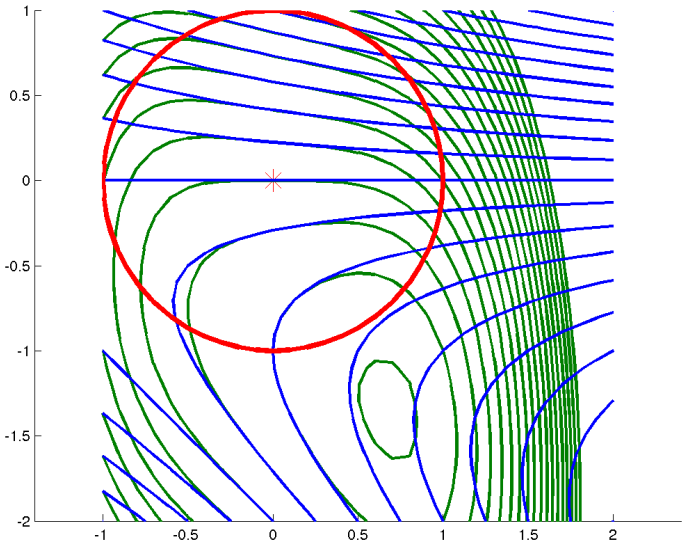
$$\min_x f(x)$$

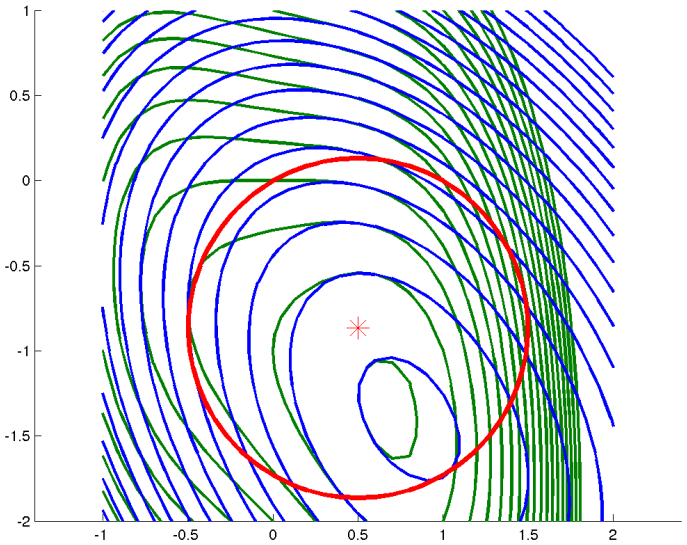
Main ingredients of solution approaches:

- Local method: given x_k (solution guess) compute a step s.
  - Gradient Descent
  - Quasi-Newton Approximation
  - Sequential Quadratic Programming
- Globalization strategy: converge from any starting point.
  - Trust region
  - Line search

Trust-Region Method

$$\begin{aligned} \min_{s} \quad & f(x_k) + s^T \nabla f(x_k) + \frac{1}{2} s^T H(x_k) s \\ \text{subject to} \quad & \|s\| \leq \Delta_k \end{aligned}$$





Trust-Region Method

1. Initialize trust-region radius
   - Constant
   - Direction
   - Interpolation

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2. Compute a new iterate
   2.1 Solve trust-region subproblem

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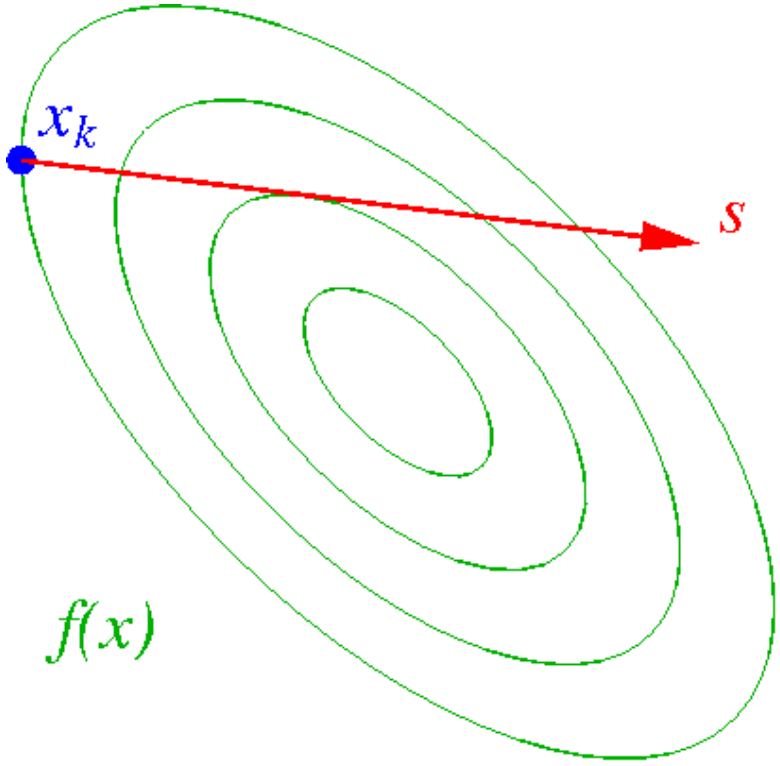
   2.2 Accept or reject iterate
   2.3 Update trust-region radius
   - Reduction
   - Interpolation
3. Check convergence

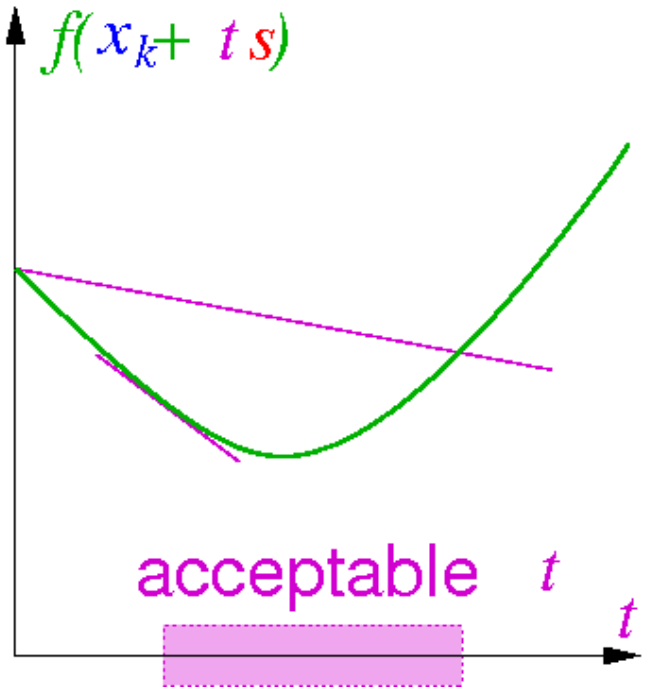
Solving the Subproblem

- Moré-Sorensen method
  - Computes global solution to subproblem
- Conjugate gradient method with trust region
  - Objective function decreases monotonically
  - Some choices need to be made
    - Preconditioner
    - Norm of direction and residual
    - Dealing with negative curvature

Line-Search Method

$$\min_{s} \quad f(x_k) + s^T \nabla f(x_k) + \frac{1}{2} s^T (H(x_k) + \lambda_k I) s$$





Line-Search Method

1. Initialize perturbation to zero
2. Solve perturbed quadratic model

$$\min_{s} \quad f(x_k) + s^T \nabla f(x_k) + \frac{1}{2} s^T (H(x_k) + \lambda_k I) s$$

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3. Find new iterate
   3.1 Search along Newton direction
   3.2 Search along gradient-based direction

Line-Search Method

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$$\min_{s} \quad f(x_k) + s^T \nabla f(x_k) + \frac{1}{2} s^T (H(x_k) + \lambda_k I) s$$

3. Find new iterate
   3.1 Search along Newton direction
   3.2 Search along gradient-based direction
4. Update perturbation
   - Decrease perturbation if the following hold
     - Iterative method succeeds
     - Search along Newton direction succeeds
   - Otherwise increase perturbation
5. Check convergence

Solving the Subproblem

- Conjugate gradient method

Solving the Subproblem

- Conjugate gradient method
- Conjugate gradient method with trust region
  - Initialize radius
    - Constant
    - Direction
    - Interpolation
  - Update radius
    - Reduction
    - Step length
    - Interpolation
  - Some choices need to be made
    - Preconditioner
    - Norm of direction and residual
    - Dealing with negative curvature

Performing the Line Search

- Backtracking Armijo Line search
  - Find t such that

$$f(x_k + ts) \leq f(x_k) + \sigma t \nabla f(x_k)^T s$$

  - Try $t = 1, \beta, \beta^2, \ldots$ for $0 < \beta < 1$
- More-Thuente Line search
  - Find t such that

$$\begin{aligned} f(x_k + ts) &\leq f(x_k) + \sigma t \nabla f(x_k)^T s \\ |\nabla f(x_k + ts)^T s| &\leq \delta |\nabla f(x_k)^T s| \end{aligned}$$

  - Construct cubic interpolant
  - Compute t to minimize interpolant
  - Refine interpolant

Updating the Perturbation

1. If increasing and $\Delta^k = 0$

$$\Delta^{k+1} = \text{Proj}_{[\ell_0, u_0]}\left(\alpha_0 \|g(x^k)\|\right)$$

2. If increasing and $\Delta^k > 0$

$$\Delta^{k+1} = \text{Proj}_{[\ell_i, u_i]}\left(\max\left(\alpha_i \|g(x^k)\|, \beta_i \Delta^k\right)\right)$$

3. If decreasing

$$\Delta^{k+1} = \min\left(\alpha_d \|g(x^k)\|, \beta_d \Delta^k\right)$$

4. If $\Delta^{k+1} < \ell_d$, then $\Delta^{k+1} = 0$

Iterative Methods

- Conjugate gradient method
  - Stop if negative curvature encountered
  - Stop if residual norm is small

Iterative Methods

- Conjugate gradient method
  - Stop if negative curvature encountered
  - Stop if residual norm is small
- Conjugate gradient method with trust region
  - Nash
    - Follow direction to boundary if first iteration
    - Stop at base of direction otherwise
  - Steihaug-Toint
    - Follow direction to boundary
  - Generalized Lanczos
    - Compute tridiagonal approximation
    - Find global solution to approximate problem on boundary
    - Initialize perturbation with approximate minimum eigenvalue

Preconditioners

- No preconditioner
- Absolute value of Hessian diagonal
- Absolute value of perturbed Hessian diagonal
- Incomplete Cholesky factorization of Hessian
- Block Jacobi with Cholesky factorization of blocks
- Scaled BFGS approximation to Hessian matrix
  - None
  - Scalar
  - Diagonal of Broyden update
  - Rescaled diagonal of Broyden update
  - Absolute value of Hessian diagonal
  - Absolute value of perturbed Hessian diagonal

Termination

- Typical convergence criteria
  - Absolute residual $\|\nabla f(x_k)\| < \tau_a$
  - Relative residual $\frac{\|\nabla f(x_k)\|}{\|\nabla f(x_k)\|} < \tau_r$
  - Unbounded objective $f(x_k) < \kappa$
  - Slow progress $|f(x_k) - f(x_{k-1})| < \epsilon$
  - Iteration limit
  - Time limit
- Solver status

```
display solve_result;           # String
display solve_result_num;       # Number
display $solve_result_table;    # Lookup table
```

Convergence Issues

- Quadratic convergence – best outcome
- Linear convergence
  - Far from a solution – $\|\nabla f(x_k)\|$ is large
  - Hessian is incorrect – disrupts quadratic convergence
  - Hessian is rank deficient – $\|\nabla f(x_k)\|$ is small
  - Limits of finite precision arithmetic
    1. $\|\nabla f(x_k)\|$ converges quadratically to small number
    2. $\|\nabla f(x_k)\|$ hovers around that number with no progress
- Domain violations such as $\frac{1}{x}$ when $x = 0$
  - Make implicit constraints explicit
- Nonglobal solution
  - Apply a multistart heuristic
  - Use global optimization solver

Some Available Software

- TRON – Newton method with trust-region
- LBFGS – Limited-memory quasi-Newton method with line search
- TAO – Toolkit for Advanced Optimization
  - NTR – Newton line-search method
  - NLS – Newton trust-region method
  - NTL – Newton line-search/trust-region method
  - LMVM – Limited-memory quasi-Newton method
  - CG – Nonlinear conjugate gradient methods

Model Formulation

- Economy with n agents and m commodities
  - $e \in \Re^{n \times m}$ are the endowments
  - $\alpha \in \Re^{n \times m}$ and $\beta \in \Re^{n \times m}$ are the utility parameters
  - $\lambda \in \Re^{n}$ are the social weights
- Social planning problem

$$\max_{x \geq 0} \quad \sum_{i=1}^{n} \lambda_i \left(\sum_{k=1}^{m} \frac{\alpha_{i,k}(1 + x_{i,k})^{1-\beta_{i,k}}}{1 - \beta_{i,k}} \right)$$

$$\text{subject to} \quad \sum_{i=1}^{n} x_{i,k} \leq \sum_{i=1}^{n} e_{i,k} \qquad \forall k = 1, \ldots, m$$

Model: social1.mod

```
param n > 0, integer;                    # Agents
param m > 0, integer;                    # Commodities

param e {1..n, 1..m} >= 0, default 1; # Endowment

param lambda {1..n} > 0;                 # Social weights
param alpha {1..n, 1..m} > 0;            # Utility parameters
param beta {1..n, 1..m} > 0;
```

Model: social1.mod

```
param n > 0, integer;                    # Agents
param m > 0, integer;                    # Commodities

param e {1..n, 1..m} >= 0, default 1; # Endowment

param lambda {1..n} > 0;                 # Social weights
param alpha {1..n, 1..m} > 0;            # Utility parameters
param beta {1..n, 1..m} > 0;


var x{1..n, 1..m} >= 0;                  # Consumption
var u{i in 1..n} =                       # Utility
  sum {k in 1..m} alpha[i,k] * (1 + x[i,k])^(1 - beta[i,k]) / (1 - beta[i,k]);
```

Model: social1.mod

```
param n > 0, integer;                    # Agents
param m > 0, integer;                    # Commodities

param e {1..n, 1..m} >= 0, default 1; # Endowment

param lambda {1..n} > 0;                 # Social weights
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var x{1..n, 1..m} >= 0;                  # Consumption
var u{i in 1..n} =                       # Utility
  sum {k in 1..m} alpha[i,k] * (1 + x[i,k])^(1 - beta[i,k]) / (1 - beta[i,k]);


maximize welfare:
    sum {i in 1..n} lambda[i] * u[i];

subject to
  consumption {k in 1..m}:
    sum {i in 1..n} x[i,k] <= sum {i in 1..n} e[i,k];
```

Data: social1.dat

```
param n := 3;           # Agents
param m := 4;           # Commodities
```

Data: social1.dat

```
param n := 3;            # Agents
param m := 4;            # Commodities


param alpha : 1    2    3    4 :=
       1      1    1    1    1
       2      1    2    3    4
       3      2    1    1    5;

param beta (tr) : 1    2    3 :=
       1         1.5   2   0.6
       2         1.6   3   0.7
       3         1.7   2   2.0
       4         1.8   2   2.5;

param : lambda :=
    1     1
    2     1
    3     1;
```

Commands: social1.cmd

```
# Load model and data
model social1.mod;
data social1.dat;

# Specify solver and options
option solver minos;
option minos_options "outlev=1";

# Solve the instance
solve;

# Output results
display x;
printf {i in 1..n} "%2d: % 5.4e\n", i, u[i];
```

Output

```
ampl: include social1.cmd;
MINOS 5.5: outlev=1
MINOS 5.5: optimal solution found.
25 iterations, objective 2.252422003
Nonlin evals: obj = 44, grad = 43.
x :=
1 1   0.0811471
1 2   0.574164
1 3   0.703454
1 4   0.267241
2 1   0.060263
2 2   0.604858
2 3   1.7239
2 4   1.47516
3 1   2.85859
3 2   1.82098
3 3   0.572645
3 4   1.2576
;

 1: -5.2111e+00
 2: -4.0488e+00
 3:  1.1512e+01
ampl: quit;
```

Model: social2.mod

```
set AGENTS;                                    # Agents
set COMMODITIES;                               # Commodities
```

Model: social2.mod

```
set AGENTS;                                     # Agents
set COMMODITIES;                                # Commodities


param e {AGENTS, COMMODITIES} >= 0, default 1; # Endowment

param lambda {AGENTS} > 0;                      # Social weights
param alpha {AGENTS, COMMODITIES} > 0;          # Utility parameters
param beta {AGENTS, COMMODITIES} > 0;

param gamma {i in AGENTS, k in COMMODITIES} := 1 - beta[i,k];
```

Model: social2.mod

```
set AGENTS;                                     # Agents
set COMMODITIES;                                # Commodities


param e {AGENTS, COMMODITIES} >= 0, default 1; # Endowment

param lambda {AGENTS} > 0;                      # Social weights
param alpha {AGENTS, COMMODITIES} > 0;          # Utility parameters
param beta {AGENTS, COMMODITIES} > 0;

param gamma {i in AGENTS, k in COMMODITIES} := 1 - beta[i,k];


var x{AGENTS, COMMODITIES} >= 0;                # Consumption
var u{i in AGENTS} =                            # Utility
  sum {k in COMMODITIES} alpha[i,k] * (1 + x[i,k])^gamma[i,k] / gamma[i,k];
```

Model: social2.mod

```
set AGENTS;                                     # Agents
set COMMODITIES;                                # Commodities



param e {AGENTS, COMMODITIES} >= 0, default 1; # Endowment

param lambda {AGENTS} > 0;                      # Social weights
param alpha {AGENTS, COMMODITIES} > 0;          # Utility parameters
param beta {AGENTS, COMMODITIES} > 0;

param gamma {i in AGENTS, k in COMMODITIES} := 1 - beta[i,k];



var x{AGENTS, COMMODITIES} >= 0;                # Consumption
var u{i in AGENTS} =                            # Utility
  sum {k in COMMODITIES} alpha[i,k] * (1 + x[i,k])^gamma[i,k] / gamma[i,k];



maximize welfare:
    sum {i in AGENTS} lambda[i] * u[i];

subject to
  consumption {k in COMMODITIES}:
    sum {i in AGENTS} x[i,k] <= sum {i in AGENTS} e[i,k];
```

Data: social2.dat

```
set COMMODITIES := Books, Cars, Food, Pens;

param: AGENTS : lambda :=
       Jorge      1
       Sven       1
       Todd       1;
```

Data: social2.dat

```
set COMMODITIES := Books, Cars, Food, Pens;

param: AGENTS : lambda :=
       Jorge      1
       Sven       1
       Todd       1;


param alpha : Books  Cars  Food  Pens :=
     Jorge      1     1     1     1
     Sven       1     2     3     4
     Todd       2     1     1     5;

param beta (tr): Jorge  Sven  Todd :=
     Books        1.5    2     0.6
     Cars         1.6    3     0.7
     Food         1.7    2     2.0
     Pens         1.8    2     2.5;
```

Commands: social2.cmd

```
# Load model and data
model social2.mod;
data social2.dat;

# Specify solver and options
option solver minos;
option minos_options "outlev=1";

# Solve the instance
solve;

# Output results
display x;
printf {i in AGENTS} "%5s: % 5.4e\n", i, u[i];
```

Output

```
ampl: include social2.cmd
MINOS 5.5: outlev=1
MINOS 5.5: optimal solution found.
25 iterations, objective 2.252422003
Nonlin evals: obj = 44, grad = 43.
x :=
Jorge Books   0.0811471
Jorge Cars    0.574164
Jorge Food    0.703454
Jorge Pens    0.267241
Sven  Books   0.060263
Sven  Cars    0.604858
Sven  Food    1.7239
Sven  Pens    1.47516
Todd  Books   2.85859
Todd  Cars    1.82098
Todd  Food    0.572645
Todd  Pens    1.2576
;

Jorge: -5.2111e+00
 Sven: -4.0488e+00
 Todd:  1.1512e+01
ampl: quit;
```

Solving Constrained Optimization Problems

$$\begin{aligned} \min_{x} \quad & f(x) \\ \text{subject to} \quad & c(x) \geq 0 \end{aligned}$$

Main ingredients of solution approaches:

- Local method: given x_k (solution guess) find a step s.
  - Sequential Quadratic Programming (SQP)
  - Sequential Linear/Quadratic Programming (SLQP)
  - Interior-Point Method (IPM)
- Globalization strategy: converge from any starting point.
  - Trust region
  - Line search
- Acceptance criteria: filter or penalty function.

Sequential Linear Programming

1. Initialize trust-region radius
2. Compute a new iterate

Sequential Linear Programming

1. Initialize trust-region radius
2. Compute a new iterate
   2.1 Solve linear program

$$\begin{aligned} \min_{s} \quad & f(x_k) + s^T \nabla f(x_k) \\ \text{subject to} \quad & c(x_k) + \nabla c(x_k)^T s \geq 0 \\ & \|s\| \leq \Delta_k \end{aligned}$$

Sequential Linear Programming

1. Initialize trust-region radius
2. Compute a new iterate
   2.1 Solve linear program

$$\begin{aligned} \min_{s} \quad & f(x_k) + s^T \nabla f(x_k) \\ \text{subject to} \quad & c(x_k) + \nabla c(x_k)^T s \geq 0 \\ & \|s\| \leq \Delta_k \end{aligned}$$

   2.2 Accept or reject iterate
   2.3 Update trust-region radius
3. Check convergence

Sequential Quadratic Programming

1. Initialize trust-region radius
2. Compute a new iterate

Sequential Quadratic Programming

1. Initialize trust-region radius
2. Compute a new iterate
   2.1 Solve quadratic program

$$\begin{aligned} \min_{s} \quad & f(x_k) + s^T \nabla f(x_k) + \frac{1}{2} s^T W(x_k) s \\ \text{subject to} \quad & c(x_k) + \nabla c(x_k)^T s \geq 0 \\ & \|s\| \leq \Delta_k \end{aligned}$$

Sequential Quadratic Programming

1. Initialize trust-region radius
2. Compute a new iterate
   2.1 Solve quadratic program

$$\begin{aligned} \min_{s} \quad & f(x_k) + s^T \nabla f(x_k) + \frac{1}{2} s^T W(x_k) s \\ \text{subject to} \quad & c(x_k) + \nabla c(x_k)^T s \geq 0 \\ & \|s\| \leq \Delta_k \end{aligned}$$

   2.2 Accept or reject iterate
   2.3 Update trust-region radius
3. Check convergence

Sequential Linear Quadratic Programming

1. Initialize trust-region radius
2. Compute a new iterate

Sequential Linear Quadratic Programming

1. Initialize trust-region radius
2. Compute a new iterate
   2.1 Solve linear program to predict active set

$$\begin{aligned} \min_{d} \quad & f(x_k) + d^T \nabla f(x_k) \\ \text{subject to} \quad & c(x_k) + \nabla c(x_k)^T d \geq 0 \\ & \|d\| \leq \Delta_k \end{aligned}$$

Sequential Linear Quadratic Programming

1. Initialize trust-region radius
2. Compute a new iterate
   2.1 Solve linear program to predict active set

$$\begin{aligned} \min_{d} \quad & f(x_k) + d^T \nabla f(x_k) \\ \text{subject to} \quad & c(x_k) + \nabla c(x_k)^T d \geq 0 \\ & \|d\| \leq \Delta_k \end{aligned}$$

   2.2 Solve equality constrained quadratic program

$$\begin{aligned} \min_{s} \quad & f(x_k) + s^T \nabla f(x_k) + \frac{1}{2} s^T W(x_k) s \\ \text{subject to} \quad & c_{\mathcal{A}}(x_k) + \nabla c_{\mathcal{A}}(x_k)^T s = 0 \end{aligned}$$

   2.3 Accept or reject iterate
   2.4 Update trust-region radius
3. Check convergence

Acceptance Criteria

- Decrease objective function value: $f(x_k + s) \leq f(x_k)$
- Decrease constraint violation: $\|c_-(x_k + s)\| \leq \|c_-(x_k)\|$

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- Decrease objective function value: $f(x_k + s) \leq f(x_k)$
- Decrease constraint violation: $\|c_-(x_k + s)\| \leq \|c_-(x_k)\|$
- Four possibilities
  1. step can decrease both $f(x)$ and $\|c_-(x)\|$ GOOD
  2. step can decrease $f(x)$ and increase $\|c_-(x)\|$???
  3. step can increase $f(x)$ and decrease $\|c_-(x)\|$???
  4. step can increase both $f(x)$ and $\|c_-(x)\|$ BAD

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- Decrease objective function value: $f(x_k + s) \leq f(x_k)$
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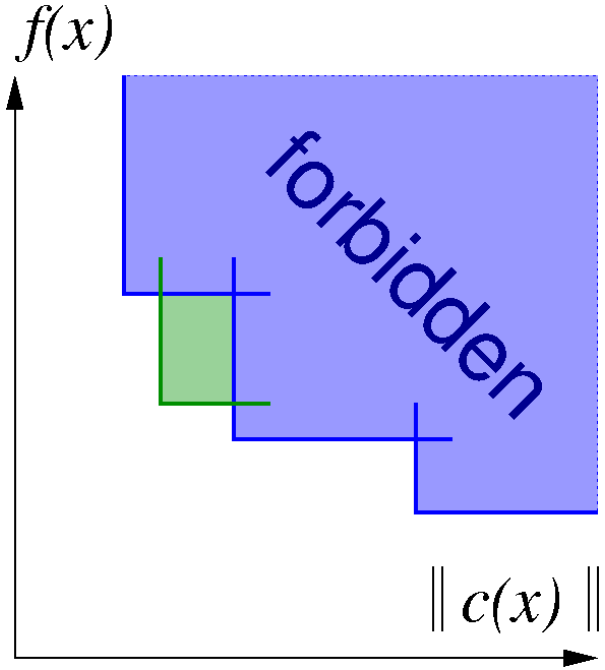
  4. step can increase both $f(x)$ and $\|c_-(x)\|$ BAD
- Filter uses concept from multi-objective optimization

(h_{k+1}, f_{k+1}) dominates (h_ℓ, f_ℓ) iff $h_{k+1} \leq h_\ell$ and $f_{k+1} \leq f_\ell$

Filter Framework

Filter $\mathcal{F}$: list of non-dominated pairs (h_ℓ, f_ℓ)

- new x_{k+1} is acceptable to filter $\mathcal{F}$ iff
  1. $h_{k+1} \leq h_\ell$ for all $\ell \in \mathcal{F}$ or
  2. $f_{k+1} \leq f_\ell$ for all $\ell \in \mathcal{F}$



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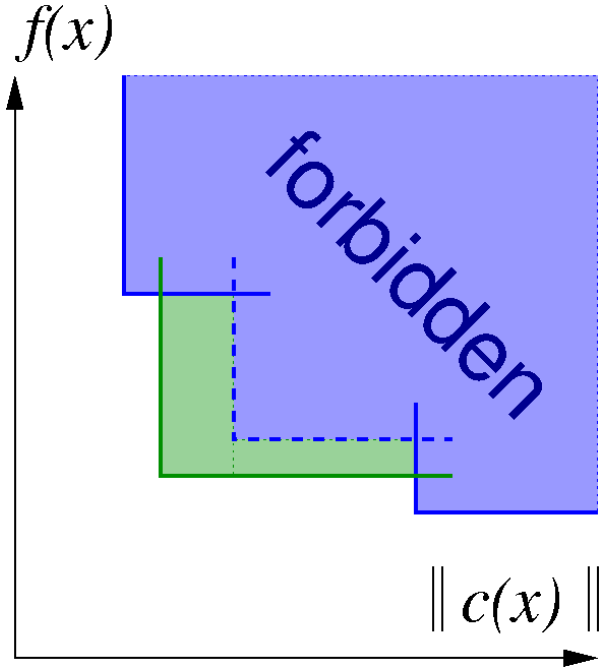
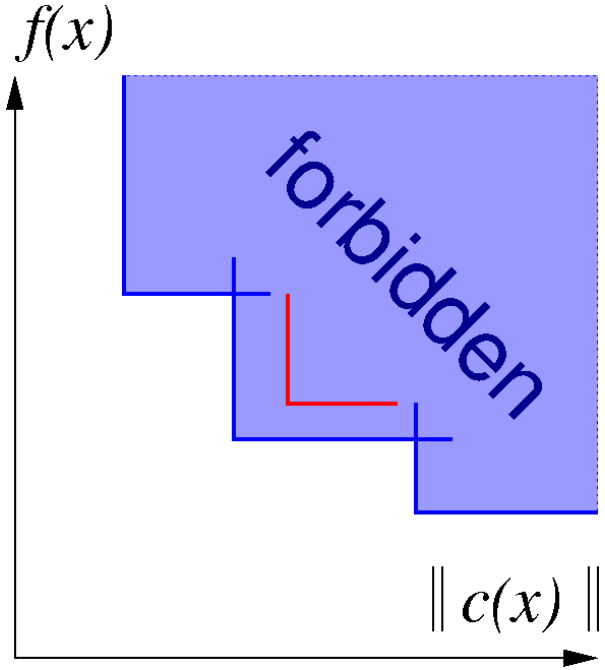
- remove redundant filter entries



Filter Framework

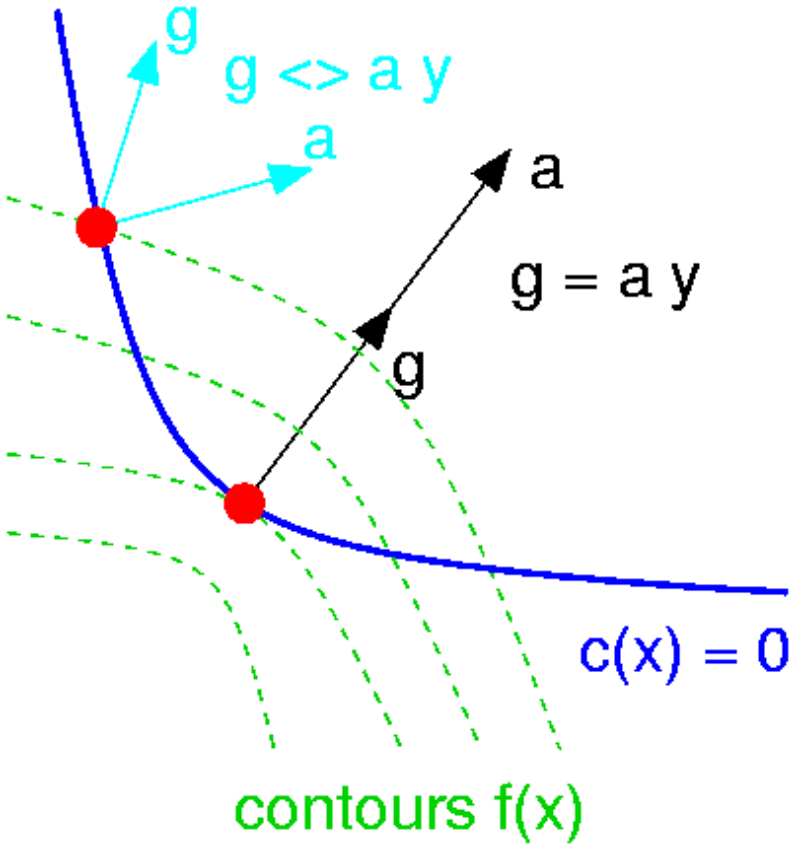
Filter $\mathcal{F}$: list of non-dominated pairs (h_ℓ, f_ℓ)

- new x_{k+1} is acceptable to filter $\mathcal{F}$ iff
  1. $h_{k+1} \leq h_\ell$ for all $\ell \in \mathcal{F}$ or
  2. $f_{k+1} \leq f_\ell$ for all $\ell \in \mathcal{F}$
- remove redundant filter entries
- new x_{k+1} is rejected if for some $\ell \in \mathcal{F}$
  1. $h_{k+1} > h_\ell$ and
  2. $f_{k+1} > f_\ell$



Convergence Criteria

- Feasible and no descent directions
  - Constraint qualification – LICQ, MFCQ
  - Linearized active constraints characterize directions
  - Objective gradient is a linear combination of constraint gradients



Optimality Conditions

- If x^* is a local minimizer and a constraint qualification holds, then there exist multipliers $\lambda^* \geq 0$ such that

$$\nabla f(x^*) - \nabla c_{\mathcal{A}}(x^*)^T \lambda^*_{\mathcal{A}} = 0$$

  - Lagrangian function $\mathcal{L}(x, \lambda) := f(x) - \lambda^T c(x)$
- Optimality conditions can be written as

$$\begin{aligned} &\nabla f(x) - \nabla c(x)^T \lambda = 0 \\ &0 \leq \lambda \quad \perp \quad c(x) \geq 0 \end{aligned}$$

  - Complementarity problem

Termination

- Feasible and complementary $\|\min(c(x_k), \lambda_k)\| \leq \tau_f$
- Optimal $\|\nabla_x \mathcal{L}(x_k, \lambda_k)\| \leq \tau_o$
- Other possible conditions
  - Slow progress
  - Iteration limit
  - Time limit
- Multipliers and reduced costs

```
display consumption.slack;      # Constraint violation
display consumption.dual;       # Lagrange multipliers

display x.rc;                   # Gradient of Lagrangian
```

Convergence Issues

- Quadratic convergence – best outcome
- Globally infeasible – linear constraints infeasible
- Locally infeasible – nonlinear constraints locally infeasible
- Unbounded objective – hard to detect
- Unbounded multipliers – constraint qualification not satisfied
- Linear convergence rate
  - Far from a solution – $\|\nabla f(x_k)\|$ is large
  - Hessian is incorrect – disrupts quadratic convergence
  - Hessian is rank deficient – $\|\nabla f(x_k)\|$ is small
  - Limits of finite precision arithmetic
- Domain violations such as $\frac{1}{x}$ when $x = 0$
  - Make implicit constraints explicit
- Nonglobal solutions
  - Apply a multistart heuristic
  - Use global optimization solver

Some Available Software

- ASTROS – Active-Set Trust-Region Optimization Solvers
- filterSQP
  - trust-region SQP; robust QP solver
  - filter to promote global convergence
- SNOPT
  - line-search SQP; null-space CG option
  - ℓ_1 exact penalty function
- SLIQUE – part of KNITRO
  - SLP-EQP
  - trust-region with ℓ_1 penalty
  - use with `knitro_options = "algorithm=3";`

Model Formulation

- Maximize discounted utility
  - $u(\cdot)$ is the utility function
  - R is the retirement age
  - T is the terminal age
  - w is the wage
  - β is the discount factor
  - r is the interest rate
- Optimization problem

$$
\begin{aligned}
\max_{s,c} \quad & \sum_{t=0}^{T} \beta^t u(c_t) \\
\text{subject to} \quad & s_{t+1} = (1+r)s_t + w - c_t \qquad t = 0, \ldots, R-1 \\
& s_{t+1} = (1+r)s_t - c_t \qquad t = R, \ldots, T \\
& s_0 = s_{T+1} = 0
\end{aligned}
$$

Model: life1.mod

```
param R > 0, integer;                    # Retirement age
param T > R, integer;                    # Terminal age

param beta >= 0, < 1;                    # Discount factor
param rate >= 0, < 1;                    # Interest rate
param wage >= 0;                         # Wage rate
```

Model: life1.mod

```
param R > 0, integer;                    # Retirement age
param T > R, integer;                    # Terminal age

param beta >= 0, < 1;                    # Discount factor
param rate >= 0, < 1;                    # Interest rate
param wage >= 0;                         # Wage rate


var c{0..T};                             # Consumption
var s{0..T+1};                           # Savings
var u{t in 0..T} = -exp(-c[t]);          # Utility
```

Model: life1.mod

```
param R > 0, integer;                    # Retirement age
param T > R, integer;                    # Terminal age

param beta >= 0, < 1;                    # Discount factor
param rate >= 0, < 1;                    # Interest rate
param wage >= 0;                         # Wage rate


var c{0..T};                             # Consumption
var s{0..T+1};                           # Savings
var u{t in 0..T} = -exp(-c[t]);          # Utility


maximize utility:
    sum {t in 0..T} beta^t * u[t];

subject to
  working {t in 0..R-1}:
    s[t+1] = (1+rate)*s[t] + wage - c[t];

  retired {t in R..T}:
    s[t+1] = (1+rate)*s[t] - c[t];

  initial:
    s[0] = 0;

  terminal:
    s[T+1] = 0;
```

Data: life1.dat

```
param R := 7;                    # Retirement age
param T := 10;                   # Terminal age

param beta := 0.9;               # Discount factor
param rate := 0.2;               # Interest rate
param wage := 1.0;               # Wage rate
```

Commands: life1.cmd

```
# Load model and data
model life1.mod;
data life1.dat;

# Specify solver and options
option solver mpec;

# Solve the instance
solve;

# Output results
printf {t in 0..T} "%2d %5.4e %5.4e\n", t, s[t], c[t] > out1.dat;
```

Output

```
ampl: include life1.cmd
AMPL interface to filter-MPEC: 20040408
: filter objective function   = -3.24322
       constraint violation = 1.01433e-11
Optimal solution found
14 iterations (0 for feasibility)
Evals: obj = 15, constr = 16, grad = 16, Hes = 15
ampl: quit;
```

Model: negishi.mod

```
set Commodities;                        # Goods and factors
set Producers within Commodities;       # Consumer goods produced
```

Model: negishi.mod

```
set Commodities;                        # Goods and factors
set Producers within Commodities;       # Consumer goods produced


##### Structure of the Nested CES Functions for Producers #####
# INodes: internal nodes for the tree in postfix order
# FUse  : factor inputs used by the node
# IUse  : internal node output used by the node
# TUse  : full list of factors referenced

set P_INodes {Producers} ordered;
set P_FUse {p in Producers, P_INodes[p]} within Commodities;
set P_IUse {p in Producers, P_INodes[p]} within P_INodes[p];
set P_TUse {p in Producers} := union {k in P_INodes[p]} P_FUse[p,k];
```

Model: negishi.mod

```
set Commodities;                        # Goods and factors
set Producers within Commodities;       # Consumer goods produced


##### Structure of the Nested CES Functions for Producers #####
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set P_INodes {Producers} ordered;
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set P_IUse {p in Producers, P_INodes[p]} within P_INodes[p];
set P_TUse {p in Producers} := union {k in P_INodes[p]} P_FUse[p,k];


##### Parameters for Producer Nested CES Functions #####

param P_Scale {p in Producers, i in P_INodes[p]}
param P_Elasticity {p in Producers, i in P_INodes[p]}
param P_FDist {p in Producers, i in P_INodes[p], k in P_FUse[p,i]}
param P_IDist {p in Producers, i in P_INodes[p], j in P_IUse[p,i]}

param P_Alpha {p in Producers, i in P_INodes[p]}
  := 1.0 - 1.0/P_Elasticity[p,i];
param P_Beta {p in Producers, i in P_INodes[p]} := 1.0 / P_Alpha[p,i];
```

Model: negishi.mod

```
##### Producer Behavior #####
# Input : quantities of commodities demanded by the producers
# Output: quantities of commodities created by the producers

var Input {p in Producers, P_TUse[p]} >= 0, := 1;

var Output {p in Producers, i in P_INodes[p]} =
  P_Scale[p,i]*(
    sum {k in P_FUse[p,i]} P_FDist[p,i,k]*Input[p,k]^P_Alpha[p,i] +
    sum {j in P_IUse[p,i]} P_IDist[p,i,j]*Output[p,j]^P_Alpha[p,i]
  )^P_Beta[p,i];
```

Model: negishi.mod

```
##### Producer Behavior #####
# Input : quantities of commodities demanded by the producers
# Output: quantities of commodities created by the producers

var Input {p in Producers, P_TUse[p]} >= 0, := 1;

var Output {p in Producers, i in P_INodes[p]} =
  P_Scale[p,i]*(
    sum {k in P_FUse[p,i]} P_FDist[p,i,k]*Input[p,k]^P_Alpha[p,i] +
    sum {j in P_IUse[p,i]} P_IDist[p,i,j]*Output[p,j]^P_Alpha[p,i]
  )^P_Beta[p,i];


##### Consumer Behavior #####
# Endow  : initial endowment of commodities
# Demand : quantities of commodities demanded by the consumer
# Utility: utility of the commodities demanded by the consumer
#        : determined from the nested CES Utility Function

param Endow {c in Consumers, k in Commodities};

var Demand {c in Consumers, C_TUse[c]} >= 0, := 1;

var Utility {c in Consumers, i in C_INodes[c]} =
  C_Scale[c,i]*(
    sum {k in C_FUse[c,i]} C_AFDist[c,i,k]*Demand[c,k]^C_Alpha[c,i] +
    sum {j in C_IUse[c,i]} C_AIDist[c,i,j]*Utility[c,j]^C_Alpha[c,i]
  )^C_Beta[c,i];
```

Model: negishi.mod

```
##### Negishi Weights #####

param Weights{c in Consumers}
  default 1 / card(Consumers);

##### Negishi Optimization Problem  #####

maximize Welfare:
  sum{c in Consumers} Weights[c]*Utility[c,last(C_INodes[c])];

subject to

Market {k in Commodities}:
  sum{c in Consumers} Endow[c,k] + (if (k in Producers) then Output[k,last(P_INodes[k])]) >=
  sum{p in Producers: k in P_TUse[p]} Input[p,k] + sum{c in Consumers: k in C_TUse[c]} Demand[c,k];
```

Data: negishi.dat

```
set Commodities :=  G1 G2 F1 F2 F3;
set Consumers := C1 C2;
set Producers := G1 G2;
set Numeraire := F1;
```

Data: negishi.dat

```
set Commodities :=  G1 G2 F1 F2 F3;
set Consumers := C1 C2;
set Producers := G1 G2;
set Numeraire := F1;


set P_INodes[G1] := I1 I2 I3;

set P_FUse[G1, I1] := F1 F2;
set P_FUse[G1, I2] := F1 F3;
set P_FUse[G1, I3] := F2 F3 G2;

set P_IUse[G1, I1] := ;
set P_IUse[G1, I2] := ;
set P_IUse[G1, I3] := I1 I2;

set P_INodes[G2] := I1 I2;

set P_FUse[G2, I1] := F1 F2;
set P_FUse[G2, I2] := F3 G1;

set P_IUse[G2, I1] := ;
set P_IUse[G2, I2] := I1;
```

Model Formulation

- Route commodities through a network
  - $\mathcal{N}$ is the set of nodes
  - $\mathcal{A} \subseteq \mathcal{N} \times \mathcal{N}$ is the set of arcs
  - $\mathcal{K}$ is the set of commodities
  - α and β are the congestion parameters
  - b denotes the supply and demand
- Multicommodity network flow problem

$$\max_{x \geq 0, f \geq 0} \quad \sum_{(i,j) \in \mathcal{A}} \left(\alpha_{i,j} f_{i,j} + \beta_{i,j} f_{i,j}^4 \right)$$

$$\text{subject to} \quad \sum_{(i,j) \in \mathcal{A}} x_{i,j,k} \leq \sum_{(j,i) \in \mathcal{A}} x_{j,i,k} + b_{i,k} \quad \forall i \in \mathcal{N}, k \in \mathcal{K}$$

$$f_{i,j} = \sum_{k \in \mathcal{K}} x_{i,j,k} \quad \forall (i,j) \in \mathcal{A}$$

Model: network.mod

```
set NODES;                                  # Nodes in network
set ARCS within NODES cross NODES;          # Arcs in network
set COMMODITIES := 1..3;                    # Commodities
```

Model: network.mod

```
set NODES;                                      # Nodes in network
set ARCS within NODES cross NODES;              # Arcs in network
set COMMODITIES := 1..3;                        # Commodities


param b {NODES, COMMODITIES} default 0;         # Supply/demand
check {k in COMMODITIES}:                       # Supply exceeds demand
  sum{i in NODES} b[i,k] >= 0;

param alpha{ARCS} >= 0;                         # Linear part
param beta{ARCS} >= 0;                          # Nonlinear part
```

Model: network.mod

```
set NODES;                                      # Nodes in network
set ARCS within NODES cross NODES;              # Arcs in network
set COMMODITIES := 1..3;                        # Commodities


param b {NODES, COMMODITIES} default 0;         # Supply/demand
check {k in COMMODITIES}:                       # Supply exceeds demand
  sum{i in NODES} b[i,k] >= 0;

param alpha{ARCS} >= 0;                         # Linear part
param beta{ARCS} >= 0;                          # Nonlinear part


var x{ARCS, COMMODITIES} >= 0;                  # Flow on arcs
var f{(i,j) in ARCS} =                          # Total flow
  sum {k in COMMODITIES} x[i,j,k];
```

Model: network.mod

```
set NODES;                                      # Nodes in network
set ARCS within NODES cross NODES;              # Arcs in network
set COMMODITIES := 1..3;                        # Commodities


param b {NODES, COMMODITIES} default 0;         # Supply/demand
check {k in COMMODITIES}:                       # Supply exceeds demand
  sum{i in NODES} b[i,k] >= 0;

param alpha{ARCS} >= 0;                         # Linear part
param beta{ARCS} >= 0;                          # Nonlinear part


var x{ARCS, COMMODITIES} >= 0;                  # Flow on arcs
var f{(i,j) in ARCS} =                          # Total flow
  sum {k in COMMODITIES} x[i,j,k];


minimize time:
  sum {(i,j) in ARCS} (alpha[i,j]*f[i,j] + beta[i,j]*f[i,j]^4);

subject to
  conserve {i in NODES, k in COMMODITIES}:
    sum {(i,j) in ARCS} x[i,j,k] <= sum{(j,i) in ARCS} x[j,i,k] + b[i,k];
```

Data: network.dat

```
set NODES := 1 2 3 4 5;

param: ARCS : alpha beta =
        1 2     1    0.5
        1 3     1    0.4
        2 3     2    0.7
        2 4     3    0.1
        3 2     1    0.0
        3 4     4    0.5
        4 1     5    0.0
        4 5     2    0.1
        5 2     0    1.0;
```

Data: network.dat

```
set NODES := 1 2 3 4 5;

param: ARCS : alpha beta =
        1 2     1    0.5
        1 3     1    0.4
        2 3     2    0.7
        2 4     3    0.1
        3 2     1    0.0
        3 4     4    0.5
        4 1     5    0.0
        4 5     2    0.1
        5 2     0    1.0;


let b[1,1] :=  7;     # Node 1, Commodity 1 supply
let b[4,1] := -7;     # Node 4, Commodity 1 demand
let b[2,2] :=  3;     # Node 2, Commodity 2 supply
let b[5,2] := -3;     # Node 5, Commodity 2 demand
let b[3,3] :=  5;     # Node 1, Commodity 3 supply
let b[1,3] := -5;     # Node 4, Commodity 3 demand

fix {i in NODES, k in COMMODITIES: (i,i) in ARCS} x[i,i,k] := 0;
```

Commands: network.cmd

```
# Load model and data
model network.mod;
data network.dat;

# Specify solver and options
option solver minos;
option minos_options "outlev=1";

# Solve the instance
solve;

# Output results
for {k in COMMODITIES} {
  printf "Commodity: %d\n", k > network.out;
  printf {(i,j) in ARCS: x[i,j,k] > 0} "%d.%d = % 5.4e\n", i, j, x[i,j,k] > network.out;
  printf "\n" > network.out;
}
```

Output

```
ampl: include network.cmd;
MINOS 5.5: outlev=1
MINOS 5.5: optimal solution found.
12 iterations, objective 1505.526478
Nonlin evals: obj = 14, grad = 13.
ampl: quit;
```

Results: network.out

```
Commodity: 1
1.2 =  3.3775e+00
1.3 =  3.6225e+00
2.4 =  6.4649e+00
3.2 =  3.0874e+00
3.4 =  5.3510e-01

Commodity: 2
2.4 =  3.0000e+00
4.5 =  3.0000e+00

Commodity: 3
3.4 =  5.0000e+00
4.1 =  5.0000e+00
```

Initial Coordinate Descent: wardrop0.cmd

```
# Load model and data
model network.mod;
data network.dat;

option solver minos;
option minos_options "outlev=1";

# Coordinate descent method
fix {(i,j) in ARCS, k in COMMODITIES} x[i,j,k];
drop {i in NODES, k in COMMODITIES} conserve[i,k];

for {iter in 1..100} {
  for {k in COMMODITIES} {
    unfix {(i,j) in ARCS} x[i,j,k];
    restore {i in NODES} conserve[i,k];

    solve;

    fix {(i,j) in ARCS} x[i,j,k];
    drop {i in NODES} conserve[i,k];
  }
}

# Output results
for {k in COMMODITIES} {
  printf "\nCommodity: %d\n", k > network.out;
  printf {(i,j) in ARCS: x[i,j,k] > 0} "%d.%d = % 5.4e\n", i, j, x[i,j,k] > network.out;
}
```

Improved Coordinate Descent: wardrop.mod

```
set NODES;                                          # Nodes in network
set ARCS within NODES cross NODES;                  # Arcs in network
set COMMODITIES := 1..3;                            # Commodities

param b {NODES, COMMODITIES} default 0;             # Supply/demand
param alpha {ARCS} >= 0;                            # Linear part
param beta {ARCS} >= 0;                             # Nonlinear part

var x {ARCS, COMMODITIES} >= 0;                     # Flow on arcs
var f {(i,j) in ARCS} =                             # Total flow
  sum {k in COMMODITIES} x[i,j,k];

minimize time {k in COMMODITIES}:
  sum {(i,j) in ARCS} (alpha[i,j]*f[i,j] + beta[i,j]*f[i,j]^4);

subject to
  conserve {i in NODES, k in COMMODITIES}:
    sum {(i,j) in ARCS} x[i,j,k] <= sum{(j,i) in ARCS} x[j,i,k] + b[i,k];

problem subprob {k in COMMODITIES}: time[k], {i in NODES} conserve[i,k],
                                    {(i,j) in ARCS} x[i,j,k], f;
```

Improved Coordinate Descent: wardrop1.cmd

```
# Load model and data
model wardrop.mod;
data wardrop.dat;

# Specify solver and options
option solver minos;
option minos_options "outlev=1";

# Coordinate descent method
for {iter in 1..100} {
  for {k in COMMODITIES} {
    solve subprob[k];
  }
}

for {k in COMMODITIES} {
  printf "Commodity: %d\n", k > wardrop.out;
  printf {(i,j) in ARCS: x[i,j,k] > 0} "%d.%d = % 5.4e\n", i, j, x[i,j,k] > wardrop.out;
  printf "\n" > wardrop.out;
}
```

Final Coordinate Descent: wardrop2.cmd

```
# Load model and data
model wardrop.mod;
data wardrop.dat;

# Specify solver and options
option solver minos;
option minos_options "outlev=1";

# Coordinate descent method
param xold{ARCS, COMMODITIES};
param xnew{ARCS, COMMODITIES};

repeat {
  for {k in COMMODITIES} {
    problem subprob[k];
    let {(i,j) in ARCS} xold[i,j,k] := x[i,j,k];
    solve;
    let {(i,j) in ARCS} xnew[i,j,k] := x[i,j,k];
  }
} until (sum {(i,j) in ARCS, k in COMMODITIES} abs(xold[i,j,k] - xnew[i,j,k]) <= 1e-6);

for {k in COMMODITIES} {
  printf "Commodity: %d\n", k > wardrop.out;
  printf {(i,j) in ARCS: x[i,j,k] > 0} "%d.%d = % 5.4e\n", i, j, x[i,j,k] > wardrop.out;
  printf "\n" > wardrop.out;
}
```

Part III

Numerical Optimization II: Optimal Control

Model Formulation

- Maximize discounted utility
  - $u(\cdot)$ is the utility function
  - R is the retirement age
  - T is the terminal age
  - w is the wage
  - β is the discount factor
  - r is the interest rate
- Optimization problem

$$
\begin{aligned}
\max_{s,c} \quad & \sum_{t=0}^{T} \beta^t u(c_t) \\
\text{subject to} \quad & s_{t+1} = (1+r)s_t + w - c_t \qquad t = 0, \ldots, R-1 \\
& s_{t+1} = (1+r)s_t - c_t \qquad t = R, \ldots, T \\
& s_0 = s_{T+1} = 0
\end{aligned}
$$

Model: life1.mod

```
param R > 0, integer;                    # Retirement age
param T > R, integer;                    # Terminal age

param beta >= 0, < 1;                    # Discount factor
param rate >= 0, < 1;                    # Interest rate
param wage >= 0;                         # Wage rate
```

Model: life1.mod

```
param R > 0, integer;                    # Retirement age
param T > R, integer;                    # Terminal age

param beta >= 0, < 1;                    # Discount factor
param rate >= 0, < 1;                    # Interest rate
param wage >= 0;                         # Wage rate


var c{0..T};                             # Consumption
var s{0..T+1};                           # Savings
var u{t in 0..T} = -exp(-c[t]);          # Utility
```

Model: life1.mod

```
param R > 0, integer;                    # Retirement age
param T > R, integer;                    # Terminal age

param beta >= 0, < 1;                    # Discount factor
param rate >= 0, < 1;                    # Interest rate
param wage >= 0;                         # Wage rate


var c{0..T};                             # Consumption
var s{0..T+1};                           # Savings
var u{t in 0..T} = -exp(-c[t]);          # Utility


maximize utility:
    sum {t in 0..T} beta^t * u[t];

subject to
  working {t in 0..R-1}:
    s[t+1] = (1+rate)*s[t] + wage - c[t];

  retired {t in R..T}:
    s[t+1] = (1+rate)*s[t] - c[t];

  initial:
    s[0] = 0;

  terminal:
    s[T+1] = 0;
```

Data: life.dat

```
param R := 75;                    # Retirement age
param T := 100;                   # Terminal age

param beta := 0.9;                # Discount factor
param rate := 0.2;                # Interest rate
param wage := 1.0;                # Wage rate
```

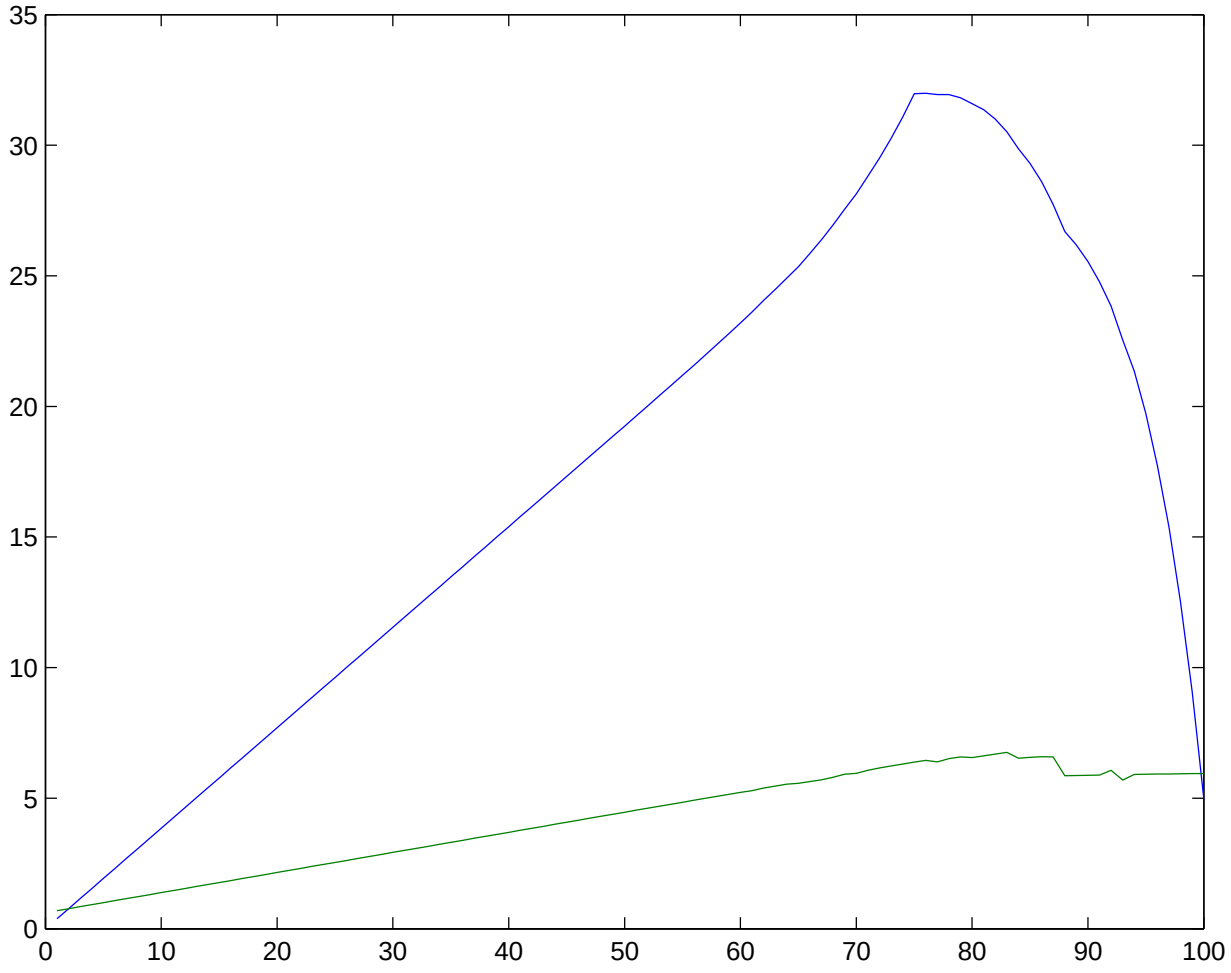
Commands: life1.cmd

```
# Load model and data
model life1.mod;
data life.dat;

# Specify solver and options
option solver mpec;

# Solve the instance
solve;

# Output results
printf {t in 0..T} "%2d %5.4e %5.4e\n", t, s[t], c[t] > out1.dat;
```

Output

```
ampl: include life1.cmd
AMPL interface to filter-MPEC: 20040408
: filter objective function   = -3.24322
       constraint violation = 1.01433e-11
Optimal solution found
14 iterations (0 for feasibility)
Evals: obj = 15, constr = 16, grad = 16, Hes = 15
ampl: quit;
```

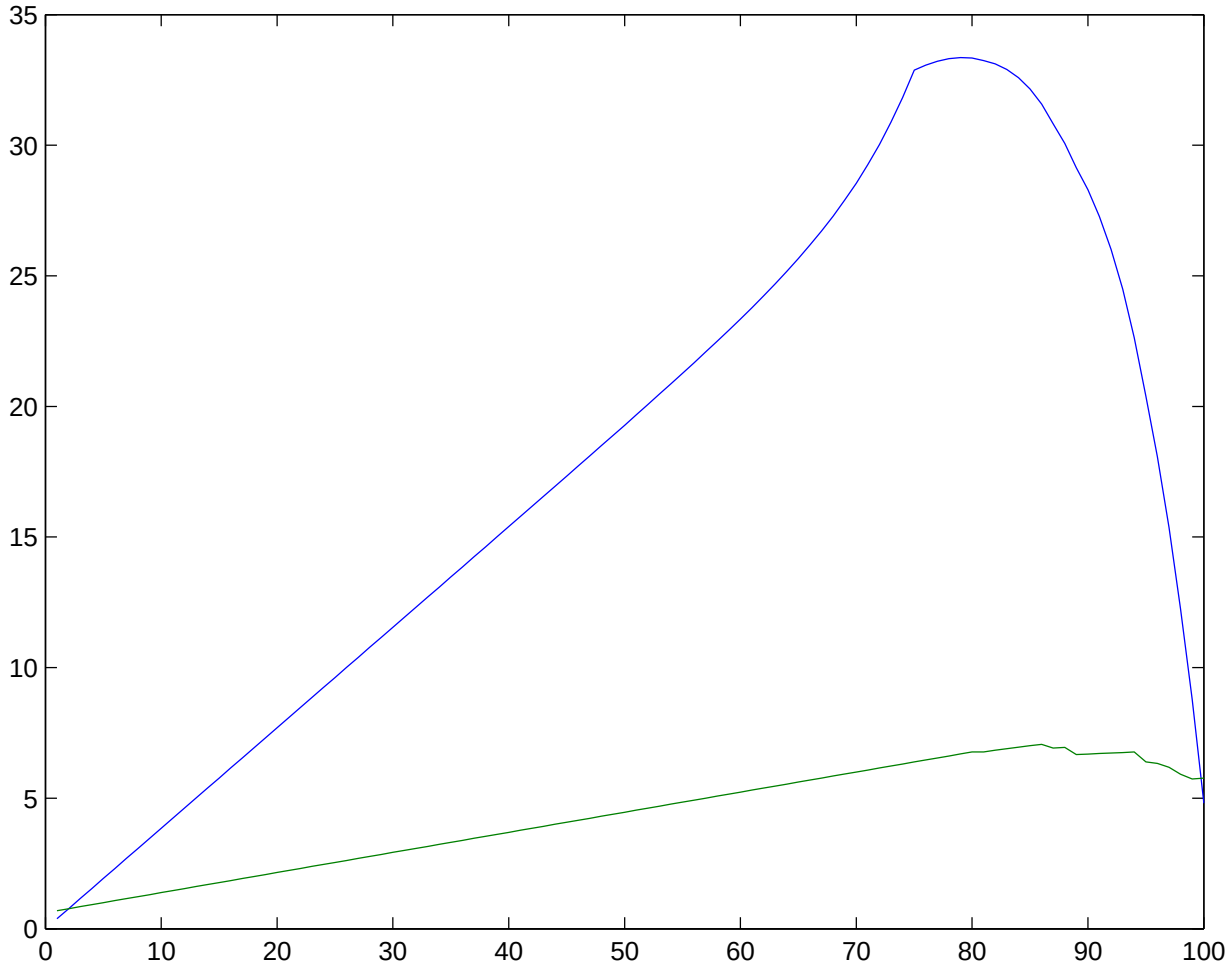
Plot of Output



Model: life2.mod

```
param R > 0, integer;                    # Retirement age
param T > R, integer;                    # Terminal age

param beta >= 0, < 1;                    # Discount factor
param rate >= 0, < 1;                    # Interest rate
param wage >= 0;                         # Wage rate

var cbar{0..T};                          # Scaled consumption
var c{t in 0..T} = cbar[t] / beta^t;     # Actual consumption
var s{0..T+1};                           # Savings
var u{t in 0..T} = -exp(-cbar[t] / beta^t);

maximize utility:
    sum {t in 0..T} beta^t * u[t];

subject to
  working {t in 0..R-1}:
    s[t+1] = (1+rate)*s[t] + wage - cbar[t] / beta^t;

  retired {t in R..T}:
    s[t+1] = (1+rate)*s[t] - cbar[t] / beta^t;

  initial:
    s[0] = 0;

  terminal:
    s[T+1] = 0;
```

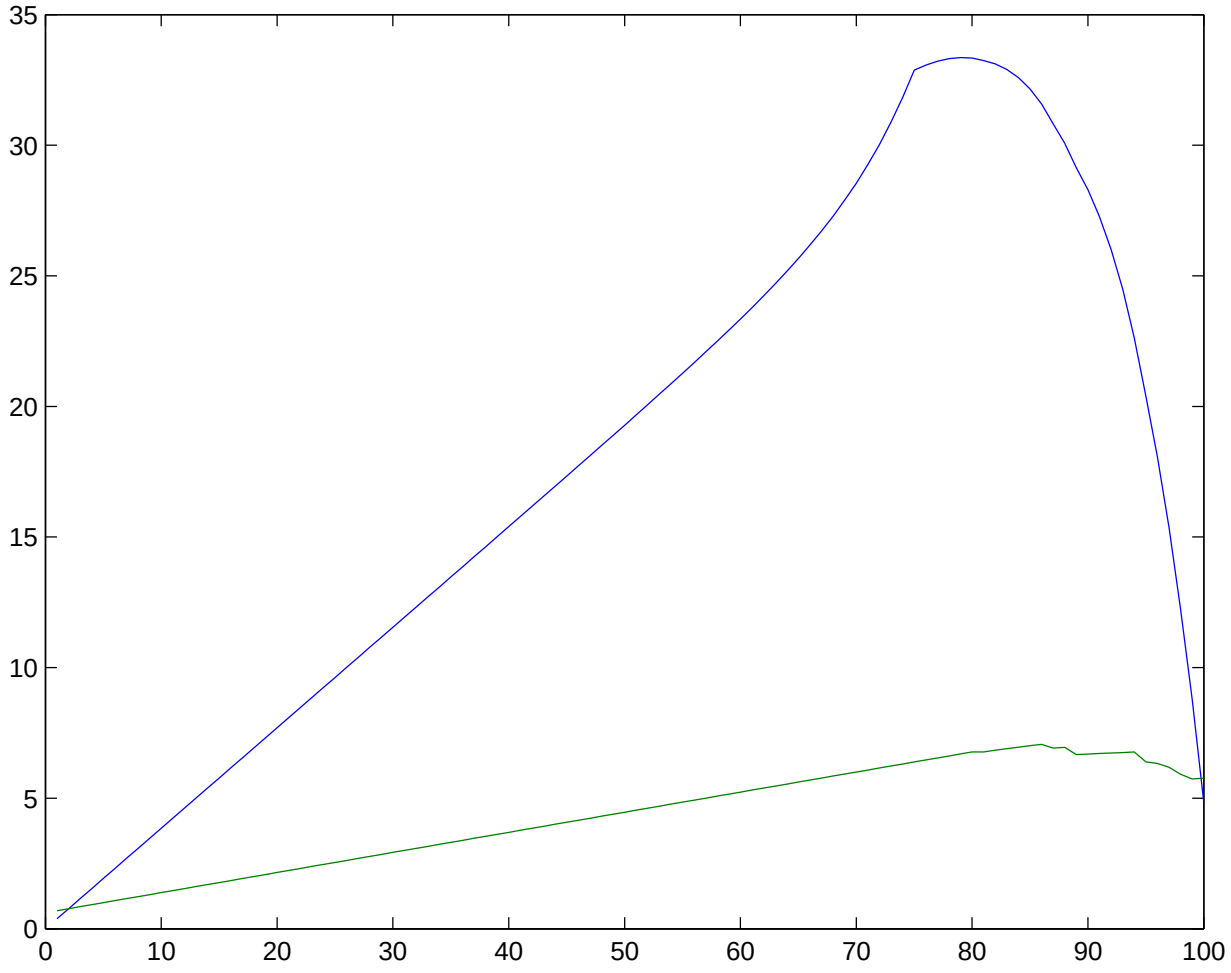
Plot of Output



Model: life3.mod

```
param R > 0, integer;                    # Retirement age
param T > R, integer;                    # Terminal age

param beta >= 0, < 1;                    # Discount factor
param rate >= 0, < 1;                    # Interest rate
param wage >= 0;                         # Wage rate

var cbar{0..T};                          # Scaled consumption
var c{t in 0..T} = cbar[t] / beta^t;     # Actual consumption
var s{0..T+1};                           # Savings
var u{t in 0..T} = -exp(-cbar[t] / beta^t);

maximize utility:
    sum {t in 0..T} beta^t * u[t];

subject to
  working {t in 0..R-1}:
    beta^t*s[t+1] = beta^t*(1+rate)*s[t] + beta^t*wage - cbar[t];

  retired {t in R..T}:
    beta^t*s[t+1] = beta^t*(1+rate)*s[t] - cbar[t];

  initial:
    s[0] = 0;

  terminal:
    s[T+1] = 0;
```

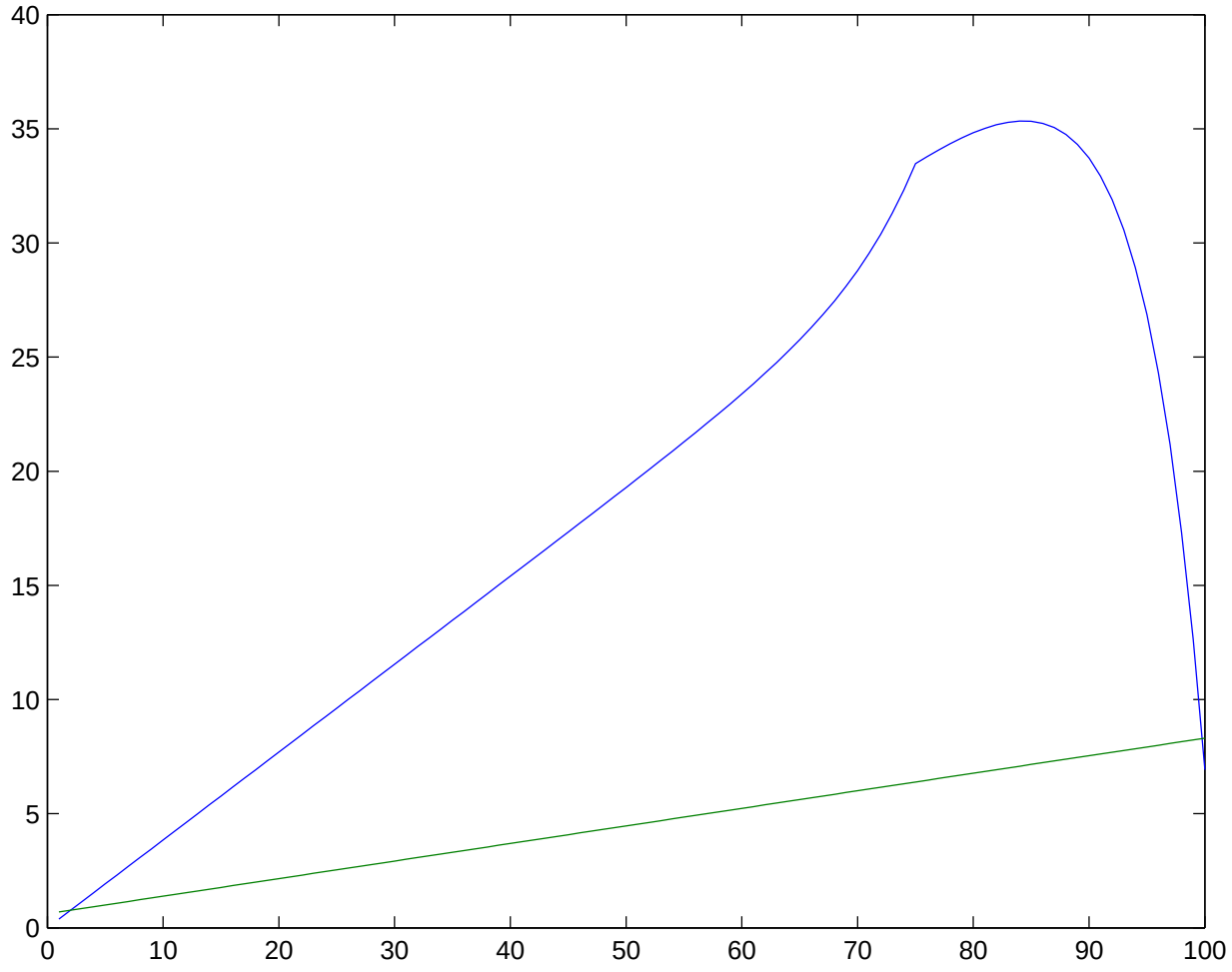
Plot of Output



Model: life4.mod

```
param R > 0, integer;                    # Retirement age
param T > R, integer;                    # Terminal age

param beta >= 0, < 1;                    # Discount factor
param rate >= 0, < 1;                    # Interest rate
param wage >= 0;                         # Wage rate

var cbar{0..T};                          # Scaled consumption
var c{t in 0..T} = cbar[t] / beta^t;     # Actual consumption
var sbar{0..T+1};                        # Scaled savings
var s{t in 0..T+1} = sbar[t] / beta^t;   # Actual savings
var u{t in 0..T} = -exp(-cbar[t] / beta^t);

maximize utility:
    sum {t in 0..T} beta^t * u[t];

subject to
  working {t in 0..R-1}:
    sbar[t+1]/beta = (1+rate)*sbar[t] + beta^t*wage - cbar[t];

  retired {t in R..T}:
    sbar[t+1]/beta = (1+rate)*sbar[t] - cbar[t];

  initial:
    sbar[0] = 0;

  terminal:
    sbar[T+1] = 0;
```

Plot of Output



Solving Constrained Optimization Problems

$$\begin{aligned} \min_{x} \quad & f(x) \\ \text{subject to} \quad & c(x) \geq 0 \end{aligned}$$

Main ingredients of solution approaches:

- Local method: given x_k (solution guess) find a step s.
  - Sequential Quadratic Programming (SQP)
  - Sequential Linear/Quadratic Programming (SLQP)
  - Interior-Point Method (IPM)
- Globalization strategy: converge from any starting point.
  - Trust region
  - Line search
- Acceptance criteria: filter or penalty function.

Interior-Point Method

- Reformulate optimization problem with slacks

$$\begin{array}{ll} \min\limits_{x} & f(x) \\ \text{subject to} & c(x) = 0 \\ & x \geq 0 \end{array}$$

- Construct perturbed optimality conditions

$$F_\tau(x, y, z) = \begin{bmatrix} \nabla f(x) - \nabla c(x)^T y - z \\ c(x) \\ Xz - \tau e \end{bmatrix}$$

- Central path $\{x(\tau), y(\tau), z(\tau) \mid \tau > 0\}$
- Apply Newton's method for sequence $\tau \searrow 0$

Interior-Point Method

1. Compute a new iterate
   1.1 Solve linear system of equations

$$\begin{bmatrix} W_k & -\nabla c(x_k)^T & -I \\ \nabla c(x_k) & 0 & 0 \\ Z_k & 0 & X_k \end{bmatrix} \begin{pmatrix} s_x \\ s_y \\ s_z \end{pmatrix} = -F_\mu(x_k, y_k, z_k)$$

   1.2 Accept or reject iterate
   1.3 Update parameters
2. Check convergence

Convergence Issues

- Quadratic convergence – best outcome
- Globally infeasible – linear constraints infeasible
- Locally infeasible – nonlinear constraints locally infeasible
- Dual infeasible – dual problem is locally infeasible
- Unbounded objective – hard to detect
- Unbounded multipliers – constraint qualification not satisfied
- Duality gap
- Domain violations such as $\frac{1}{x}$ when $x = 0$
  - Make implicit constraints explicit
- Nonglobal solutions
  - Apply a multistart heuristic
  - Use global optimization solver

Some Available Software

- IPOPT – open source in COIN-OR
  - line-search filter algorithm
- KNITRO
  - trust-region Newton to solve barrier problem
  - ℓ_1 penalty barrier function
  - Newton system: direct solves or null-space CG
- LOQO
  - line-search method
  - Newton system: modified Cholesky factorization

Optimal Technology

Optimize energy production schedule and transition between old and new reduced-carbon technology to meet carbon targets

- Maximize social welfare
- Constraints
  - Limit total greenhouse gas emissions
  - Low-carbon technology less costly as it becomes widespread
- Assumptions on emission rates, economic growth, and energy costs

Model Formulation

- Finite time: $t \in [0, T]$
- Instantaneous energy output: $q^o(t)$ and $q^n(t)$
- Cumulative energy output: $x^o(t)$ and $x^n(t)$

$$x^n(t) = \int_0^t q^n(\tau) d\tau$$

- Discounted greenhouse gases emissions

$$\int_0^T e^{-at} \left(b_o q^o(t) + b_n q^n(t) \right) dt \leq z_T$$

- Consumer surplus $S(Q(t), t)$ derived from utility
- Production costs
  - c_o per unit cost of old technology
  - $c_n(x^n(t))$ per unit cost of new technology (learning by doing)

Continuous-Time Model

$$\max_{\{q^o,q^n,x^n,z\}(t)} \quad \int_0^T e^{-rt}\left[S(q^o(t)+q^n(t),t) - c_o q^o(t) - c_n(x^n(t))q^n(t)\right] dt$$

subject to

$$\dot{x^n}(t) = q^n(t) \quad x(0) = x_0 = 0$$

$$\dot{z}(t) = e^{-at}\left(b_o q^o(t) + b_n q^n(t)\right) \quad z(0) = z_0 = 0$$

$$z(T) \leq z_T$$

$$q^o(t) \geq 0, \quad q^n(t) \geq 0.$$

Optimal Technology Penetration

Discretization:

- $t \in [0, T]$ replaced by $N + 1$ equally spaced points $t_i = ih$
- $h := T/N$ time integration step-length
- approximate $q_i^n \simeq q^n(t_i)$ etc.

Replace differential equation

$$\dot{x}(t) = q^n(t)$$

by

$$x_{i+1} = x_i + hq_i^n$$

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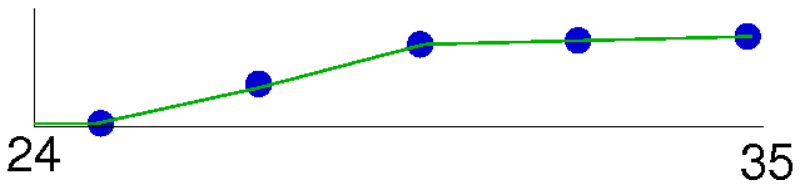
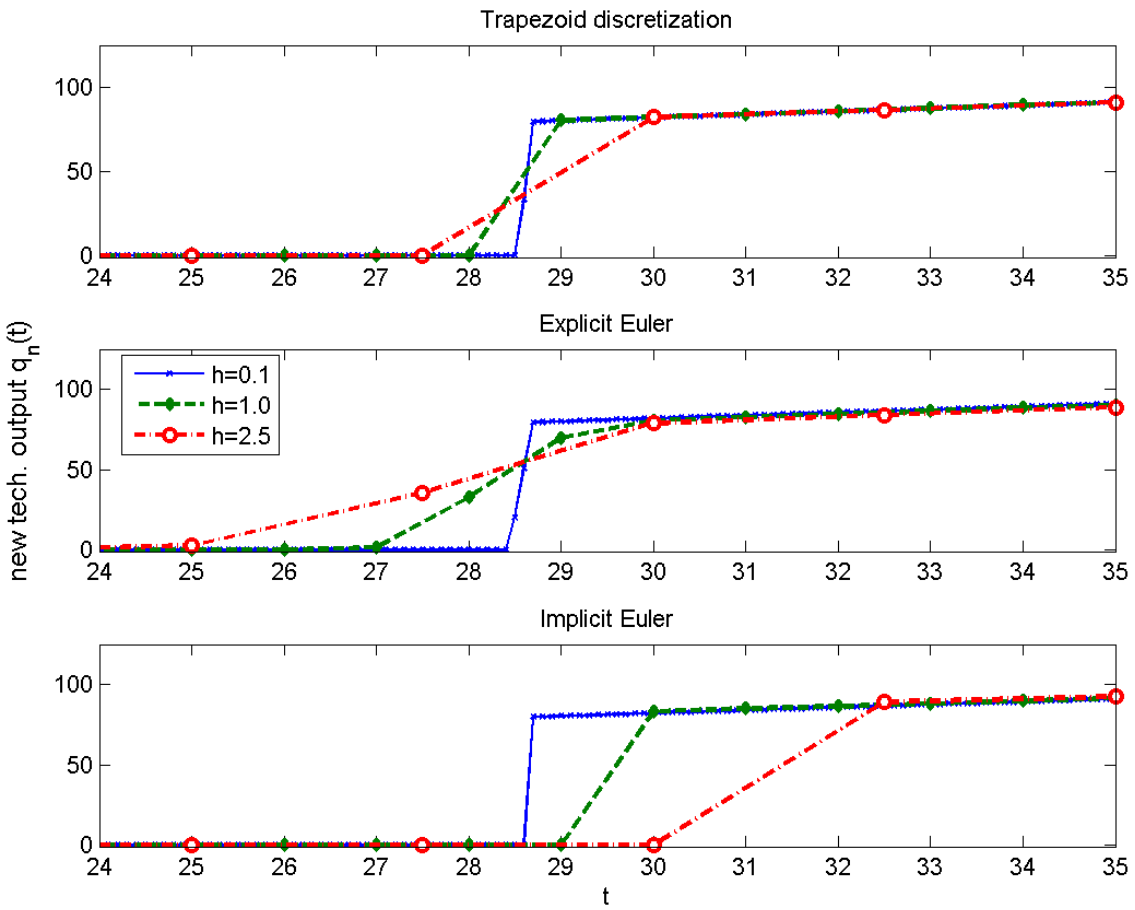


Output of new technology between $t = 24$ and $t = 35$

Solution with Varying h



Output for different discretization schemes and step-sizes

Optimal Technology Penetration

Add adjustment cost to model building of capacity:

Capital and Investment:

- $K^j(t)$ amount of capital in technology j at t.
- $I^j(t)$ investment to increase $K^j(t)$.
- initial capital level as $\bar{K}_0^j$:

Notation:

- $Q(t) = q^o(t) + q^n(t)$
- $C(t) = C^o(q^o(t), K^o(t)) + C^n(q^n(t), K^n(t))$
- $I(t) = I^o(t) + I^n(t)$
- $K(t) = K^o(t) + K^n(t)$

Optimal Technology Penetration

$$\underset{\{q^j,K^j,I^j,x,z\}(t)}{\text{maximize}} \quad \left\{ \int_0^T e^{-rt} \left[\tilde{S}(Q(t),t) - C(t) - K(t) \right] dt + e^{-rT} K(T) \right\}$$

subject to

$$\dot{x}(t) = q^n(t), \quad x(0) = x_0 = 0$$

$$\dot{K}^j(t) = -\delta K^j(t) + I^j(t), \quad K^j(0) = \bar{K}_0^j, \quad j \in \{o, n\}$$

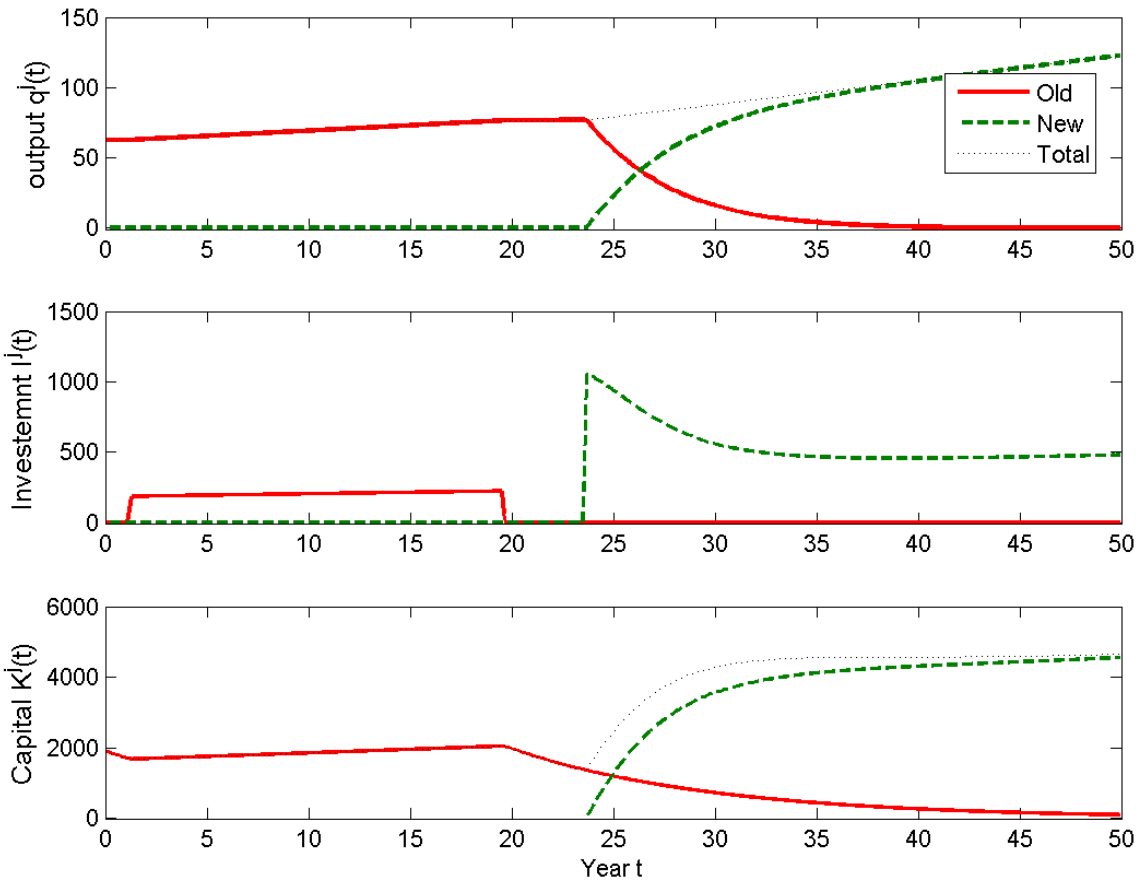
$$\dot{z}(t) = e^{-at}[b_o q^o(t) + b_n q^n(t)], \quad z(0) = z_0 = 0$$

$$z(T) \leq z_T$$

$$q^j(t) \geq 0, \; j \in \{o, n\}$$

$$I^j(t) \geq 0, \; j \in \{o, n\}$$

Optimal Technology Penetration



Optimal output, investment, and capital for 50% CO2 reduction.

Pitfalls of Discretizations [Hager, 2000]

Optimal Control Problem

$$\text{minimize } \frac{1}{2}\int_0^1 u^2(t) + 2y^2(t)dt$$

subject to

$$\begin{aligned} \dot{y}(t) &= \tfrac{1}{2}y(t) + u(t),\ t \in [0,1], \\ y(0) &= 1. \end{aligned}$$

$$\Rightarrow y^*(t) = \frac{2e^{3t} + e^3}{e^{3t/2}(2 + e^3)},$$
$$u^*(t) = \frac{2(e^{3t} - e^3)}{e^{3t/2}(2 + e^3)}.$$

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Discretize with 2nd order RK

$$\text{minimize } \frac{h}{2}\sum_{k=0}^{K-1} u_{k+1/2}^2 + 2y_{k+1/2}^2$$

subject to $(k = 0, \ldots, K)$:

$$\begin{aligned} y_{k+1/2} &= y_k + \frac{h}{2}(\tfrac{1}{2}y_k + u_k), \\ y_{k+1} &= y_k + h(\tfrac{1}{2}y_{k+1/2} + u_{k+1/2}) \end{aligned}$$

Pitfalls of Discretizations [Hager, 2000]

Optimal Control Problem

$$\text{minimize } \tfrac{1}{2}\int_0^1 u^2(t) + 2y^2(t)dt$$

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Discretize with 2nd order RK

$$\text{minimize } \frac{h}{2}\sum_{k=0}^{K-1} u_{k+1/2}^2 + 2y_{k+1/2}^2$$

subject to $(k = 0, \ldots, K)$:

$$\begin{aligned} y_{k+1/2} &= y_k + \frac{h}{2}(\tfrac{1}{2}y_k + u_k), \\ y_{k+1} &= y_k + h(\tfrac{1}{2}y_{k+1/2} + u_{k+1/2}) \end{aligned}$$

Discrete solution $(k = 0, \ldots, K)$:

$$y_k = 1, \quad y_{k+1/2} = 0,$$
$$u_k = -\frac{4+h}{2h}, \quad u_{k+1/2} = 0,$$

Tips to Solve Continuous-Time Problems

- Use discretize-then-optimize with different schemes
- Refine discretization: $h = 1$ discretization is nonsense
- Check implied discretization of adjoints

Tips to Solve Continuous-Time Problems

- Use discretize-then-optimize with different schemes
- Refine discretization: $h = 1$ discretization is nonsense
- Check implied discretization of adjoints

Alternative: Optimize-Then-Discretize

- Consistent adjoint/dual discretization
- Discretized gradients can be wrong!
- Harder for inequality constraints

Ordered Sets

```
param V, integer;                                # Number of vertices
param E, integer;                                # Number of elements

set VERTICES := {1..V};                          # Vertex indices
set ELEMENTS := {1..E};                          # Element indices
set COORDS   := {1..3} ordered;                  # Spatial coordinates

param T{ELEMENTS, 1..4} in VERTICES;             # Tetrahedral elements

var x{VERTICES, COORDS};                         # Position of vertices

var norm{e in ELEMENTS} = sum{i in COORDS, j in 1..4}
  (x[T[e,j], i] - x[T[e,1], i])^2;

var area{e in ELEMENTS} = sum{i in COORDS}
  (x[T[e,2], i] - x[T[e,1], i]) *
    ((x[T[e,3], nextw(i)] - x[T[e,1], nextw(i)]) *
     (x[T[e,4], prevw(i)] - x[T[e,1], prevw(i)]) -
     (x[T[e,3], prevw(i)] - x[T[e,1], prevw(i)]) *
     (x[T[e,4], nextw(i)] - x[T[e,1], nextw(i)]));

minimize f: sum {e in ELEMENTS} norm[e] / max(area[e], 0) ^ (2 / 3);
```

Circular Sets

```
param V, integer;                                # Number of vertices
param E, integer;                                # Number of elements

set VERTICES := {1..V};                          # Vertex indices
set ELEMENTS := {1..E};                          # Element indices
set COORDS   := {1..3} circular;                 # Spatial coordinates

param T{ELEMENTS, 1..4} in VERTICES;             # Tetrahedral elements

var x{VERTICES, COORDS};                         # Position of vertices

var norm{e in ELEMENTS} = sum{i in COORDS, j in 1..4}
  (x[T[e,j], i] - x[T[e,1], i])^2;

var area{e in ELEMENTS} = sum{i in COORDS}
  (x[T[e,2], i] - x[T[e,1], i]) *
    ((x[T[e,3], next(i)] - x[T[e,1], next(i)]) *
     (x[T[e,4], prev(i)] - x[T[e,1], prev(i)]) -
     (x[T[e,3], prev(i)] - x[T[e,1], prev(i)]) *
     (x[T[e,4], next(i)] - x[T[e,1], next(i)]));

minimize f: sum {e in ELEMENTS} norm[e] / max(area[e], 0) ^ (2 / 3);
```

Part IV

Numerical Optimization III: Complementarity Constraints

Nash Games

- Non-cooperative game played by n individuals
  - Each player selects a strategy to optimize their objective
  - Strategies for the other players are fixed
- Equilibrium reached when no improvement is possible

Nash Games

- Non-cooperative game played by n individuals
  - Each player selects a strategy to optimize their objective
  - Strategies for the other players are fixed
- Equilibrium reached when no improvement is possible
- Characterization of two player equilibrium (x^*, y^*)

$$x^* \in \begin{cases} \arg\min\limits_{x \geq 0} & f_1(x, y^*) \\ \text{subject to} & c_1(x) \leq 0 \end{cases}$$

$$y^* \in \begin{cases} \arg\min\limits_{y \geq 0} & f_2(x^*, y) \\ \text{subject to} & c_2(y) \leq 0 \end{cases}$$

Nash Games

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$$y^* \in \begin{cases} \arg\min\limits_{y \geq 0} & f_2(x^*, y) \\ \text{subject to} & c_2(y) \leq 0 \end{cases}$$

- Many applications in economics
  - Bimatrix games
  - Cournot duopoly models
  - General equilibrium models
  - Arrow-Debreau models

Complementarity Formulation

- Assume each optimization problem is convex
  - $f_1(\cdot, y)$ is convex for each y
  - $f_2(x, \cdot)$ is convex for each x
  - $c_1(\cdot)$ and $c_2(\cdot)$ satisfy constraint qualification
- Then the first-order conditions are necessary and sufficient

$$\begin{array}{ll} \min\limits_{x \geq 0} & f_1(x, y^*) \\ \text{subject to} & c_1(x) \leq 0 \end{array} \quad \Leftrightarrow \quad \begin{array}{lcl} 0 \leq x & \perp & \nabla_x f_1(x, y^*) + \lambda_1^T \nabla_x c_1(x) \geq 0 \\ 0 \leq \lambda_1 & \perp & -c_1(x) \geq 0 \end{array}$$

Complementarity Formulation

- Assume each optimization problem is convex
  - $f_1(\cdot, y)$ is convex for each y
  - $f_2(x, \cdot)$ is convex for each x
  - $c_1(\cdot)$ and $c_2(\cdot)$ satisfy constraint qualification
- Then the first-order conditions are necessary and sufficient

$$\begin{array}{ll} \min\limits_{y \geq 0} & f_2(x^*, y) \\ \text{subject to} & c_2(y) \leq 0 \end{array} \quad \Leftrightarrow \quad \begin{array}{lcl} 0 \leq y & \perp & \nabla_y f_2(x^*, y) + \lambda_2^T \nabla_y c_2(y) \geq 0 \\ 0 \leq \lambda_2 & \perp & -c_2(y) \geq 0 \end{array}$$

Complementarity Formulation

- Assume each optimization problem is convex
  - $f_1(\cdot, y)$ is convex for each y
  - $f_2(x, \cdot)$ is convex for each x
  - $c_1(\cdot)$ and $c_2(\cdot)$ satisfy constraint qualification
- Then the first-order conditions are necessary and sufficient

$$\begin{array}{lcl} 0 \leq x & \perp & \nabla_x f_1(x,y) + \lambda_1^T \nabla_x c_1(x) \geq 0 \\ 0 \leq y & \perp & \nabla_y f_2(x,y) + \lambda_2^T \nabla_y c_2(y) \geq 0 \\ 0 \leq \lambda_1 & \perp & -c_1(y) \geq 0 \\ 0 \leq \lambda_2 & \perp & -c_2(y) \geq 0 \end{array}$$

- Nonlinear complementarity problem
  - Square system – number of variables and constraints the same
  - Each solution is an equilibrium for the Nash game

Model Formulation

- Economy with n agents and m commodities
  - $e \in \Re^{n \times m}$ are the endowments
  - $\alpha \in \Re^{n \times m}$ and $\beta \in \Re^{n \times m}$ are the utility parameters
  - $p \in \Re^{m}$ are the commodity prices
- Agent i maximizes utility with budget constraint

$$\begin{aligned} \max_{x_{i,*} \geq 0} \quad & \sum_{k=1}^{m} \frac{\alpha_{i,k}(1 + x_{i,k})^{1-\beta_{i,k}}}{1 - \beta_{i,k}} \\ \text{subject to} \quad & \sum_{k=1}^{m} p_k \left(x_{i,k} - e_{i,k} \right) \leq 0 \end{aligned}$$

- Market k sets price for the commodity

$$0 \leq p_k \quad \perp \quad \sum_{i=1}^{n} \left(e_{i,k} - x_{i,k} \right) \geq 0$$

Model: cge.mod

```
set AGENTS;                                     # Agents
set COMMODITIES;                                # Commodities

param e {AGENTS, COMMODITIES} >= 0, default 1; # Endowment

param alpha {AGENTS, COMMODITIES} > 0;          # Utility parameters
param beta {AGENTS, COMMODITIES} > 0;

var x {AGENTS, COMMODITIES};                    # Consumption (no bounds!)
var l {AGENTS};                                 # Multipliers (no bounds!)
var p {COMMODITIES};                            # Prices (no bounds!)

var du {i in AGENTS, k in COMMODITIES} =        # Marginal prices
  alpha[i,k] / (1 + x[i,k])^beta[i,k];

subject to
  optimality {i in AGENTS, k in COMMODITIES}:
    0 <= x[i,k] complements -du[i,k] + p[k] * l[i] >= 0;

  budget {i in AGENTS}:
    0 <= l[i]   complements sum {k in COMMODITIES} p[k]*(e[i,k] - x[i,k]) >= 0;

  market {k in COMMODITIES}:
    0 <= p[k]   complements sum {i in AGENTS} (e[i,k] - x[i,k]) >= 0;
```

Data: cge.dat

```
set AGENTS := Jorge, Sven, Todd;
set COMMODITIES := Books, Cars, Food, Pens;

param alpha : Books  Cars  Food  Pens :=
     Jorge      1     1     1     1
     Sven       1     2     3     4
     Todd       2     1     1     5;

param beta (tr): Jorge  Sven  Todd :=
     Books        1.5    2     0.6
     Cars         1.6    3     0.7
     Food         1.7    2     2.0
     Pens         1.8    2     2.5;
```

Commands: cge.cmd

```
# Load model and data
model cge.mod;
data cge.dat;

# Specify solver and options
option presolve 0;
option solver "pathampl";

# Solve the instance
solve;

# Output results
printf {i in AGENTS, k in COMMODITIES} "%5s %5s: % 5.4e\n", i, k, x[i,k] > cge.out;
printf "\n" > cge.out;
printf {k in COMMODITIES} "%5s: % 5.4e\n", k, p[k] > cge.out;
```

Results: cge.out

```
Jorge Books:  8.9825e-01
Jorge  Cars:  1.4651e+00
Jorge  Food:  1.2021e+00
Jorge  Pens:  6.8392e-01
 Sven Books:  2.5392e-01
 Sven  Cars:  7.2054e-01
 Sven  Food:  1.6271e+00
 Sven  Pens:  1.4787e+00
 Todd Books:  1.8478e+00
 Todd  Cars:  8.1431e-01
 Todd  Food:  1.7081e-01
 Todd  Pens:  8.3738e-01

Books:  1.0825e+01
 Cars:  6.6835e+00
 Food:  7.3983e+00
 Pens:  1.1081e+01
```

Commands: cgenum.cmd

```
# Load model and data
model cge.mod;
data cge.dat;

# Specify solver and options
option presolve 0;
option solver "pathampl";

# Solve the instance
drop market['Books'];
fix p['Books'] := 1;
solve;

# Output results
printf {i in AGENTS, k in COMMODITIES} "%5s %5s: % 5.4e\n", i, k, x[i,k] > cgenum.out;
printf "\n" > cgenum.out;
printf {k in COMMODITIES} "%5s: % 5.4e\n", k, p[k] > cgenum.out;
```

Results: cgenum.out

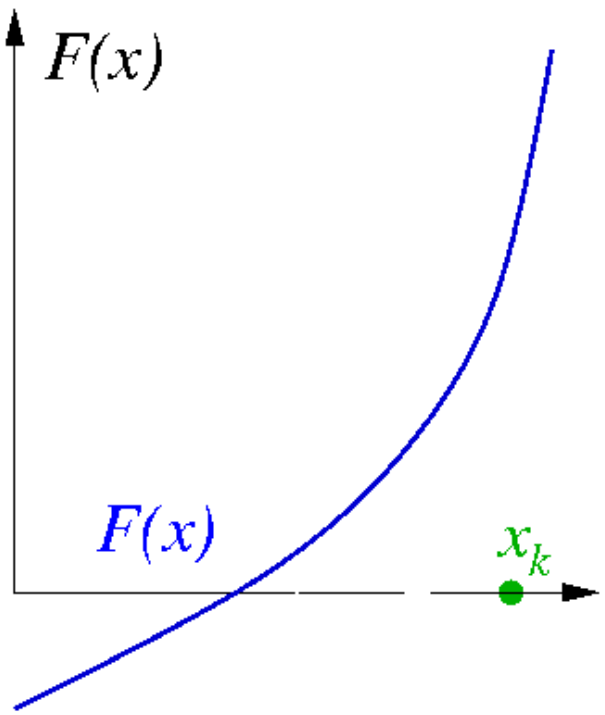
```
Jorge Books:  8.9825e-01
Jorge  Cars:  1.4651e+00
Jorge  Food:  1.2021e+00
Jorge  Pens:  6.8392e-01
 Sven Books:  2.5392e-01
 Sven  Cars:  7.2054e-01
 Sven  Food:  1.6271e+00
 Sven  Pens:  1.4787e+00
 Todd Books:  1.8478e+00
 Todd  Cars:  8.1431e-01
 Todd  Food:  1.7081e-01
 Todd  Pens:  8.3738e-01

Books:  1.0000e+00
 Cars:  6.1742e-01
 Food:  6.8345e-01
 Pens:  1.0237e+00
```

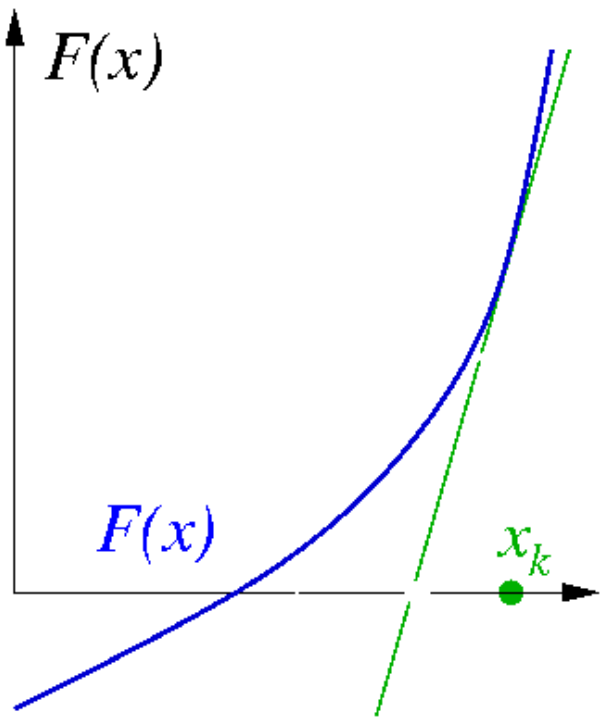
Pitfalls

- Nonsquare systems
  - Side variables
  - Side constraints
- Orientation of equations
  - Skew symmetry preferred
  - Proximal point perturbation
- AMPL presolve
  - `option presolve 0;`

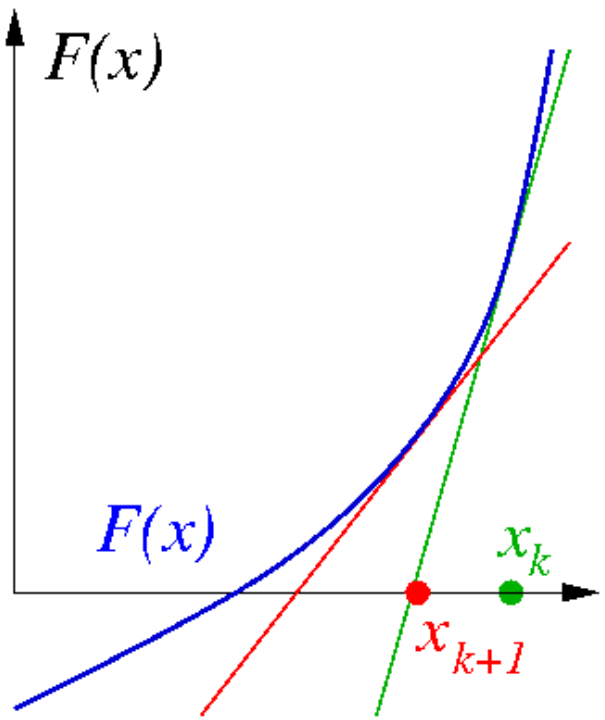
Newton Method for Nonlinear Equations



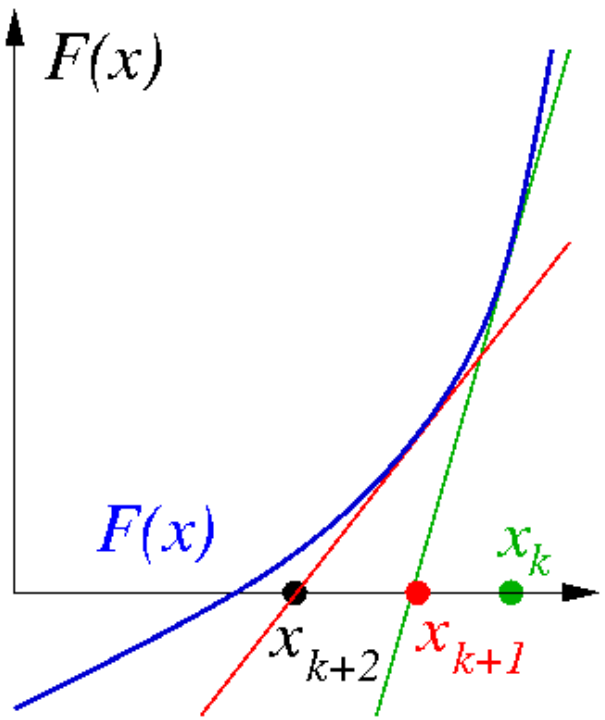
Newton Method for Nonlinear Equations



Newton Method for Nonlinear Equations



Newton Method for Nonlinear Equations



Methods for Complementarity Problems

- Sequential linearization methods (PATH)
  1. Solve the linear complementarity problem

$$0 \leq x \quad \perp \quad F(x_k) + \nabla F(x_k)(x - x_k) \geq 0$$

  2. Perform a line search along merit function
  3. Repeat until convergence

Methods for Complementarity Problems

- Sequential linearization methods (PATH)
  1. Solve the linear complementarity problem

$$0 \leq x \quad \perp \quad F(x_k) + \nabla F(x_k)(x - x_k) \geq 0$$

  2. Perform a line search along merit function
  3. Repeat until convergence
- Semismooth reformulation methods (SEMI)
  - Solve linear system of equations to obtain direction
  - Globalize with a trust region or line search
  - Less robust in general
- Interior-point methods

Semismooth Reformulation

- Define Fischer-Burmeister function

$$\phi(a,b) := a + b - \sqrt{a^2 + b^2}$$

  - $\phi(a,b) = 0$ iff $a \geq 0$, $b \geq 0$, and $ab = 0$
- Define the system

$$[\Phi(x)]_i = \phi(x_i, F_i(x))$$

  - x^* solves complementarity problem iff $\Phi(x^*) = 0$
- Nonsmooth system of equations

Semismooth Algorithm

1. Calculate $H^k \in \partial_B \Phi(x^k)$ and solve the following system for d^k:

$$H^k d^k = -\Phi(x^k)$$

If this system either has no solution, or

$$\nabla \Psi(x^k)^T d^k \leq -p_1 \|d^k\|^{p_2}$$

is not satisfied, let $d^k = -\nabla \Psi(x^k)$.

Semismooth Algorithm

1. Calculate $H^k \in \partial_B \Phi(x^k)$ and solve the following system for d^k:

$$H^k d^k = -\Phi(x^k)$$

   If this system either has no solution, or

$$\nabla \Psi(x^k)^T d^k \leq -p_1 \|d^k\|^{p_2}$$

   is not satisfied, let $d^k = -\nabla \Psi(x^k)$.

2. Compute smallest nonnegative integer i^k such that

$$\Psi(x^k + \beta^{i^k} d^k) \leq \Psi(x^k) + \sigma \beta^{i^k} \nabla \Psi(x^k) d^k$$

3. Set $x^{k+1} = x^k + \beta^{i^k} d^k$, $k = k + 1$, and go to 1.

Convergence Issues

- Quadratic convergence – best outcome
- Linear convergence
  - Far from a solution – $r(x_k)$ is large
  - Jacobian is incorrect – disrupts quadratic convergence
  - Jacobian is rank deficient – $\|\nabla r(x_k)\|$ is small
  - Converge to local minimizer – guarantees rank deficiency
  - Limits of finite precision arithmetic
    1. $r(x_k)$ converges quadratically to small number
    2. $r(x_k)$ hovers around that number with no progress
- Domain violations such as $\frac{1}{x}$ when $x = 0$

Some Available Software

- PATH – sequential linearization method
- MILES – sequential linearization method
- SEMI – semismooth linesearch method
- TAO – Toolkit for Advanced Optimization
  - SSLS – full-space semismooth linesearch methods
  - ASLS – active-set semismooth linesearch methods
  - RSCS – reduced-space method

Definition

- Leader-follower game
  - Dominant player (leader) selects a strategy y^*
  - Then followers respond by playing a Nash game

$$x_i^* \in \begin{cases} \arg\min\limits_{x_i \geq 0} & f_i(x, y) \\ \text{subject to} & c_i(x_i) \leq 0 \end{cases}$$

- Leader solves optimization problem with equilibrium constraints

$$\begin{array}{ll} \min\limits_{y \geq 0, x, \lambda} & g(x, y) \\ \text{subject to} & h(y) \leq 0 \\ & 0 \leq x_i \quad \perp \quad \nabla_{x_i} f_i(x, y) + \lambda_i^T \nabla_{x_i} c_i(x_i) \geq 0 \\ & 0 \leq \lambda_i \quad \perp \quad -c_i(x_i) \geq 0 \end{array}$$

- Many applications in economics
  - Optimal taxation
  - Tolling problems

Model Formulation

- Economy with n agents and m commodities
  - $e \in \Re^{n \times m}$ are the endowments
  - $\alpha \in \Re^{n \times m}$ and $\beta \in \Re^{n \times m}$ are the utility parameters
  - $p \in \Re^{m}$ are the commodity prices
- Agent i maximizes utility with budget constraint

$$\begin{aligned} \max_{x_{i,*} \geq 0} \quad & \sum_{k=1}^{m} \frac{\alpha_{i,k}(1 + x_{i,k})^{1-\beta_{i,k}}}{1 - \beta_{i,k}} \\ \text{subject to} \quad & \sum_{k=1}^{m} p_k \left(x_{i,k} - e_{i,k} \right) \leq 0 \end{aligned}$$

- Market k sets price for the commodity

$$0 \leq p_k \quad \perp \quad \sum_{i=1}^{n} \left(e_{i,k} - x_{i,k} \right) \geq 0$$

Model: cgempec.mod

```
set LEADER;                                     # Leader
set FOLLOWERS;                                  # Followers
set AGENTS := LEADER union FOLLOWERS;           # All the agents
check: (card(LEADER) == 1 && card(LEADER inter FOLLOWERS) == 0);

set COMMODITIES;                                # Commodities

param e {AGENTS, COMMODITIES} >= 0, default 1; # Endowment

param alpha {AGENTS, COMMODITIES} > 0;          # Utility parameters
param beta {AGENTS, COMMODITIES} > 0;

var x {AGENTS, COMMODITIES};                    # Consumption (no bounds!)
var l {FOLLOWERS};                              # Multipliers (no bounds!)
var p {COMMODITIES};                            # Prices (no bounds!)

var u {i in AGENTS} =                           # Utility
  sum {k in COMMODITIES} alpha[i,k] * (1 + x[i,k])^(1 - beta[i,k]) / (1 - beta[i,k]);
var du {i in AGENTS, k in COMMODITIES} =        # Marginal prices
  alpha[i,k] / (1 + x[i,k])^beta[i,k];
```

Model: cgempec.mod

```
maximize
  objective: sum {i in LEADER} u[i];

subject to
  leader_budget {i in LEADER}:
    sum {k in COMMODITIES} p[k]*(e[i,k] - x[i,k]) >= 0;

  optimality {i in FOLLOWERS, k in COMMODITIES}:
    0 <= x[i,k] complements -du[i,k] + p[k] * l[i] >= 0;

  budget {i in FOLLOWERS}:
    0 <= l[i]   complements sum {k in COMMODITIES} p[k]*(e[i,k] - x[i,k]) >= 0;

  market {k in COMMODITIES}:
    0 <= p[k]   complements sum {i in AGENTS} (e[i,k] - x[i,k]) >= 0;
```

Data: cgempec.dat

```
set LEADER := Jorge;
set FOLLOWERS := Sven, Todd;
set COMMODITIES := Books, Cars, Food, Pens;

param alpha : Books  Cars  Food  Pens :=
     Jorge      1     1     1     1
     Sven       1     2     3     4
     Todd       2     1     1     5;

param beta (tr): Jorge  Sven  Todd :=
     Books        1.5    2     0.6
     Cars         1.6    3     0.7
     Food         1.7    2     2.0
     Pens         1.8    2     2.5;
```

Commands: cgempec.cmd

```
# Load model and data
model cgempec.mod;
data cgempec.dat;

# Specify solver and options
option presolve 0;
option solver "loqo";

# Solve the instance
drop market['Books'];
fix p['Books'] := 1;
solve;

# Output results
printf {i in AGENTS, k in COMMODITIES} "%5s %5s: % 5.4e\n", i, k, x[i,k] > cgempec.out;
printf "\n" > cgempec.out;
printf {k in COMMODITIES} "%5s: % 5.4e\n", k, p[k] > cgempec.out;
```

Output: cgempec.out

```
     Stackleberg                          Nash Game
Jorge Books:  9.2452e-01          Jorge Books:  8.9825e-01
Jorge  Cars:  1.3666e+00          Jorge  Cars:  1.4651e+00
Jorge  Food:  1.1508e+00          Jorge  Food:  1.2021e+00
Jorge  Pens:  7.7259e-01          Jorge  Pens:  6.8392e-01
 Sven Books:  2.5499e-01           Sven Books:  2.5392e-01
 Sven  Cars:  7.4173e-01           Sven  Cars:  7.2054e-01
 Sven  Food:  1.6657e+00           Sven  Food:  1.6271e+00
 Sven  Pens:  1.4265e+00           Sven  Pens:  1.4787e+00
 Todd Books:  1.8205e+00           Todd Books:  1.8478e+00
 Todd  Cars:  8.9169e-01           Todd  Cars:  8.1431e-01
 Todd  Food:  1.8355e-01           Todd  Food:  1.7081e-01
 Todd  Pens:  8.0093e-01           Todd  Pens:  8.3738e-01

Books:  1.0000e+00                Books:  1.0000e+00
 Cars:  5.9617e-01                 Cars:  6.1742e-01
 Food:  6.6496e-01                 Food:  6.8345e-01
 Pens:  1.0700e+00                 Pens:  1.0237e+00
```

Nonlinear Programming Formulation

$$
\begin{aligned}
\min_{x,y,\lambda,s,t\geq 0} \quad & g(x,y) \\
\text{subject to} \quad & h(y) \leq 0 \\
& s_i = \nabla_{x_i} f_i(x,y) + \lambda_i^T \nabla_{x_i} c_i(x_i) \\
& t_i = -c_i(x_i) \\
& \sum_i \left(s_i^T x_i + \lambda_i t_i\right) \leq 0
\end{aligned}
$$

- Constraint qualification fails
  - Lagrange multiplier set unbounded
  - Constraint gradients linearly dependent
  - Central path does not exist
- Able to prove convergence results for some methods
- Reformulation very successful and versatile in practice

Penalization Approach

$$
\begin{aligned}
\min_{x,y,\lambda,s,t\geq 0} \quad & g(x,y) + \pi \sum_i \left(s_i^T x_i + \lambda_i t_i \right) \\
\text{subject to} \quad & h(y) \leq 0 \\
& s_i = \nabla_{x_i} f_i(x,y) + \lambda_i^T \nabla_{x_i} c_i(x_i) \\
& t_i = -c_i(x_i)
\end{aligned}
$$

- Optimization problem satisfies constraint qualification
- Need to increase π

Relaxation Approach

$$
\begin{aligned}
\min_{x,y,\lambda,s,t\geq 0} \quad & g(x,y) \\
\text{subject to} \quad & h(y) \leq 0 \\
& s_i = \nabla_{x_i} f_i(x,y) + \lambda_i^T \nabla_{x_i} c_i(x_i) \\
& t_i = -c_i(x_i) \\
& \sum_i \left(s_i^T x_i + \lambda_i t_i \right) \leq \tau
\end{aligned}
$$

- Need to decrease τ

Limitations

- Multipliers may not exist
- Solvers can have a hard time computing solutions
  - Try different algorithms
  - Compute feasible starting point
- Stationary points may have descent directions
  - Checking for descent is an exponential problem
  - Strong stationary points found in certain cases
- Many stationary points – global optimization

Limitations

- Multipliers may not exist
- Solvers can have a hard time computing solutions
  - Try different algorithms
  - Compute feasible starting point
- Stationary points may have descent directions
  - Checking for descent is an exponential problem
  - Strong stationary points found in certain cases
- Many stationary points – global optimization
- Formulation of follower problem
  - Multiple solutions to Nash game
  - Nonconvex objective or constraints
  - Existence of multipliers

Model Formulation

- Firm $f \in \mathcal{F}$ chooses output x_f to maximize profit
  - u is the utility function

$$u = \left(1 + \sum_{f \in \mathcal{F}} x_f^{\alpha}\right)^{\frac{\eta}{\alpha}}$$

  - α and η are parameters
  - c_f is the unit cost for each firm
- In particular, for each firm $f \in \mathcal{F}$

$$x_f^* \quad \in \quad \arg\max_{x_f \geq 0} \left(\frac{\partial u}{\partial x_f} - c_f\right) x_f$$

- First-order optimality conditions

$$0 \leq x_f \quad \perp c_f - \frac{\partial u}{\partial x_f} - x_f \frac{\partial^2 u}{\partial x_f^2} \geq 0$$

Model: oligopoly.mod

```
set FIRMS;                                          # Firms in problem

param c {FIRMS};                                    # Unit cost
param alpha > 0;                                    # Constants
param eta > 0;

var x {FIRMS} default 0.1;                          # Output (no bounds!)

var s = 1 + sum {f in FIRMS} x[f]^alpha;            # Summation term
var u = s^(eta/alpha);                              # Utility
var du {f in FIRMS} =                               # Marginal price
  eta * s^(eta/alpha - 1) * x[f]^(alpha - 1);
var dudu {f in FIRMS} =                             # Derivative
  eta * (eta - alpha) * s^(eta/alpha - 2) * x[f]^(2 * alpha - 2) +
  eta * (alpha - 1  ) * s^(eta/alpha - 1) * x[f]^(    alpha - 2);

compl {f in FIRMS}:
  0 <= x[f] complements c[f] - du[f] - x[f] * dudu[f] >= 0;
```

Data: oligopoly.dat

```
param: FIRMS :  c :=
         1     0.07
         2     0.08
         3     0.09;

param alpha := 0.999;
param eta := 0.2;
```

Commands: oligopoly.cmd

```
# Load model and data
model oligopoly.mod;
data oligopoly.dat;

# Specify solver and options
option presolve 0;
option solver "pathampl";

# Solve complementarity problem
solve;

# Output the results
printf {f in FIRMS} "Output for firm %2d: % 5.4e\n", f, x[f] > oligcomp.out;
```

Results: oligopoly.out

```
Output for firm  1:  8.3735e-01
Output for firm  2:  5.0720e-01
Output for firm  3:  1.7921e-01
```

Model Formulation

- Players select strategies to minimize loss
  - $p \in \Re^n$ is the probability player 1 chooses each strategy
  - $q \in \Re^m$ is the probability player 2 chooses each strategy
  - $A \in \Re^{n \times m}$ is the loss matrix for player 1
  - $B \in \Re^{n \times m}$ is the loss matrix for player 2
- Optimization problem for player 1

$$\begin{array}{ll} \min\limits_{0 \le p \le 1} & p^T A q \\ \text{subject to} & e^T p = 1 \end{array}$$

- Optimization problem for player 2

$$\begin{array}{ll} \min\limits_{0 \le q \le 1} & p^T B q \\ \text{subject to} & e^T q = 1 \end{array}$$

Model Formulation

- Players select strategies to minimize loss
  - $p \in \Re^n$ is the probability player 1 chooses each strategy
  - $q \in \Re^m$ is the probability player 2 chooses each strategy
  - $A \in \Re^{n \times m}$ is the loss matrix for player 1
  - $B \in \Re^{n \times m}$ is the loss matrix for player 2
- Complementarity problem

$$\begin{array}{lcl} 0 \leq p \leq 1 & \perp & Aq - \lambda_1 \\ 0 \leq q \leq 1 & \perp & B^T p - \lambda_2 \\ \lambda_1 \text{ free} & \perp & e^T p = 1 \\ \lambda_2 \text{ free} & \perp & e^T q = 1 \end{array}$$

Model: bimatrix1.mod

```
param n > 0, integer;           # Strategies for player 1
param m > 0, integer;           # Strategies for player 2

param A{1..n, 1..m};            # Loss matrix for player 1
param B{1..n, 1..m};            # Loss matrix for player 2

var p{1..n};                    # Probability player 1 selects strategy i
var q{1..m};                    # Probability player 2 selects strategy j
var lambda1;                    # Multiplier for constraint
var lambda2;                    # Multiplier for constraint

subject to
  opt1 {i in 1..n}:             # Optimality conditions for player 1
    0 <= p[i] <= 1 complements sum{j in 1..m} A[i,j] * q[j] - lambda1;

  opt2 {j in 1..m}:             # Optimality conditions for player 2
    0 <= q[j] <= 1 complements sum{i in 1..n} B[i,j] * p[i] - lambda2;

  con1:
    lambda1 complements sum{i in 1..n} p[i] = 1;

  con2:
    lambda2 complements sum{j in 1..m} q[j] = 1;
```

Model: bimatrix2.mod

```
param n > 0, integer;           # Strategies for player 1
param m > 0, integer;           # Strategies for player 2

param A{1..n, 1..m};            # Loss matrix for player 1
param B{1..n, 1..m};            # Loss matrix for player 2

var p{1..n};                    # Probability player 1 selects strategy i
var q{1..m};                    # Probability player 2 selects strategy j
var lambda1;                    # Multiplier for constraint
var lambda2;                    # Multiplier for constraint

subject to
  opt1 {i in 1..n}:             # Optimality conditions for player 1
    0 <= p[i] complements sum{j in 1..m} A[i,j] * q[j] - lambda1 >= 0;

  opt2 {j in 1..m}:             # Optimality conditions for player 2
    0 <= q[j] complements sum{i in 1..n} B[i,j] * p[i] - lambda2 >= 0;

  con1:
    0 <= lambda1 complements sum{i in 1..n} p[i] >= 1;

  con2:
    0 <= lambda2 complements sum{j in 1..m} q[j] >= 1;
```

Model: bimatrix3.mod

```
param n > 0, integer;            # Strategies for player 1
param m > 0, integer;            # Strategies for player 2

param A{1..n, 1..m};             # Loss matrix for player 1
param B{1..n, 1..m};             # Loss matrix for player 2

var p{1..n};                     # Probability player 1 selects strategy i
var q{1..m};                     # Probability player 2 selects strategy j

subject to
  opt1 {i in 1..n}:              # Optimality conditions for player 1
    0 <= p[i] complements sum{j in 1..m} A[i,j] * q[j] >= 1;

  opt2 {j in 1..m}:              # Optimality conditions for player 2
    0 <= q[j] complements sum{i in 1..n} B[i,j] * p[i] >= 1;
```

Part V

Numerical Optimization IV: Extensions

Global Optimization

I need to find the GLOBAL minimum!

- use any NLP solver (often work well!)
- use the multi-start trick from previous slides
- global optimization based on branch-and-reduce: `BARON`
  - constructs global underestimators
  - refines region by branching
  - tightens bounds by solving LPs
  - solve problems with 100s of variables
- "voodoo" solvers: genetic algorithm & simulated annealing
  no convergence theory ... usually worse than deterministic

Derivative-Free Optimization

My model does not have derivatives!

- Change your model ... good models have derivatives!
- pattern-search methods for $\min f(x)$
  - evaluate $f(x)$ at stencil $x_k + \Delta M$
  - move to new best point
  - extend to NLP; some convergence theory h
  - matlab: `NOMADm.m`; parallel `APPSPACK`
- solvers based on building interpolating quadratic models
  - DFO project on `www.coin-or.org`
  - Mike Powell's `NEWUOA` quadratic model
- "voodoo" solvers: genetic algorithm & simulated annealing
  no convergence theory ... usually worse than deterministic

Optimization with Integer Variables

Mixed-Integer Nonlinear Program (MINLP)

- modeling discrete choices $\Rightarrow 0 - 1$ variables
- modeling integer decisions $\Rightarrow$ integer variables
  e.g. number of different stocks in portfolio (8-10)
  not number of beers sold at Goose Island (millions)

MINLP solvers:

- branch (separate $z_i = 0$ and $z_i = 1$) and cut
- solve millions of NLP relaxations: `MINLPBB`, `SBB`
- outer approximation: iterate MILP and NLP solvers
  `BONMIN (COIN-OR)` & `FilMINT` on NEOS

Portfolio Management

- N: Universe of asset to purchase
- x_i: Amount of asset i to hold
- B: Budget

$$\text{minimize } u(x) \quad \text{subject to } \sum_{i \in N} x_i = B, \quad x \geq 0$$

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- Markowitz: $u(x) \stackrel{\text{def}}{=} -\alpha^T x + \lambda x^T Q x$
  - α: maximize expected returns
  - Q: variance-covariance matrix of expected returns
  - λ: minimize risk; aversion parameter

More Realistic Models

- $b \in \mathbb{R}^{|N|}$ of "benchmark" holdings
- Benchmark Tracking: $u(x) \stackrel{\text{def}}{=} (x-b)^T Q (x-b)$
  - Constraint on $\mathbb{E}$[Return]: $\alpha^T x \geq r$

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- Limit Names: $|i \in N \ : \ x_i > 0| \leq K$
  - Use binary indicator variables to model the implication $x_i > 0 \Rightarrow y_i = 1$
  - Implication modeled with variable upper bounds:

$$x_i \leq B y_i \qquad \forall i \in N$$

  - $\sum_{i \in N} y_i \leq K$

Optimization Conclusions

Optimization is General Modeling Paradigm

- linear, nonlinear, equations, inequalities
- integer variables, equilibrium, control

AMPL (GAMS) Modeling and Programming Languages

- express optimization problems
- use automatic differentiation
- easy access to state-of-the-art solvers

Optimization Software

- open-source: COIN-OR, IPOPT, SOPLEX, & ASTROS (soon)
- current solver limitations on laptop:
  - 1,000,000 variables/constraints for LPs
  - 100,000 variables/constraints for NLPs/NCPs
  - 100 variables/constraints for global optimization
  - 500,000,000 variable LP on BlueGene/P