Closed-Loop Equilibrium in a Multi-Stage Innovation Race

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1. Introduction

In recent years there have been many efforts to rigorously model innovation processes and competitions. The early work of Kamien and Schwartz, summarized in Kamien and Schwartz (1982), concentrated on the decision-theoretic problems associated with innovtion and lead to the study of equilibrium of competition in innovation contained in Loury (1979), Lee and Wilde (1980), Reinganum (1982a,b), and Dasgupta and Stiglitz (1980a,b). These analyses examined one-shot innovation processes—as long as no competitor won, all competitors were equal. Also, they assumed that there was just one available innovation technology.

This paper has two purposes. First, we examine the equilibrium of a race for a prize where each of two agents controls independent R&D projects. At each moment, both agents work to advance his own state of knowledge while knowing that of his opponent. The race ends when one of the firms has achieved a critical state of knowledge, here called "success", which results in some social gain, a portion of which is the winner's prize. This model is intended to be a stylized representation of a multi-stage R&D race where the competitors choose a portfolio of innovation investments while observing his opponent's position.[1] Such a model can address questions concerning each player's reaction to his rival's advances and the resulting allocation of resources across alternative innovation approaches of varying risk. We characterize the equilibrium of the resulting stochastic game.

Second, we use approximation techniques to more precisely examine the nature of the subgame-perfect equilibrium of our game. Global closed-form solutions to our

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[1] Section 2 compares our model with the multiperiod models of Fudenberg, Gilbert, Stiglitz, and Tirole (1983), Lee (1982), Telser (1982), and Harris and Vickers (1985).

general model are not known and likely do not exit. The approximation techniques used below can provide answers to interesting questions for some open set of games. While such an approach does not yield a global resolution of the issues, it does provide guidance as to what is possibly true globally and what are the critical factors. The presentation of this analysis is itself a second independent purpose of the paper since it represents a general way to analyze subgame-perfect equilibria of dynamic games without imposing economically unmotivated restrictions on the functional forms of critical model elements.

More specifically, we find that if the prize to the innovator and the net social benefits are "small" (in a sense specified below) the model yields several results. First, if the prize equals the benefits, there is excessive innovation effort, a result common to innovation models of this nature. Second, since agents can be at differing levels of knowledge in our model, we would like to compare the relative efficiency of resource allocation across firms. We find that lagging firms are less efficient in that if there is to be a momentary subsidy of innovation effort, the first dollars of such a subsidy should go to the leading firm. Third, in spite of the relative inefficiency of the lagging firm, it is optimal to let competition continue unitl some firm enjoys complete success. Fourth, in spite of the excess innovation effort, it is optimal to set the prize nearly equal to the social benefit.

Fifth, since agents choose how to allocate resources across projects of varying riskiness, we examine the allocative efficiency of investment within firms. We find that there is relatively excessive investment in the riskier projects. Sixth, a strategic feature of much interest is the nature of the reactions of each innovator to the other's advances in knowledge. We find that if one player advances, the other will surely increase its effort in risky projects, a movement contrary to the socially optimal reaction, but may increase or decrease effort in less risky projects.

Some of our results hold because the multi-stage nature of the game disappears if the net social benefits is small. However, other features, particularly the nature of players' reactions and the risk allaocation decisions, are related critically to the multi-stage subgame-perfect nature of our analysis. This indicates that we have successfully peeked into the nature of subgame-perfect equilibrium in innovation races. Furthermore, we indicate how other approximations could be carried out, showing that the viability of this approach does not rely on the small prize. The only thing which is needed for the application of the approximation techniques used below is some example with a known closed-form solution. These demonstrations are conducted with sufficient generality that it is apparent that the approach to closed-loop subgame perfect equilibrium analysis we take is not specific to this model and therefore of general interest in game-theoretic analysis of dynamic strategic interaction.

Section 2 describes the general model. Section 3 gives an overview of the approximation technique which we utilize below and section 4 demonstrates it in detail for a useful special case. We then examine the nature of our problem for the case of a small net social value, discussing in section 5 the social optimum and in section 6 the competitive outcome. Section 7 compares the optimum and equilibrium outcomes and section 8 examines some implications for optimal social policy given rivalrous innovation. Section 9 discusses the relation of our analysis with other approaches, arguing that our approach gives a method to generalize solutions to problems which generate closed-form solutions. Section 10 concludes.

2. **The Model**

We will investigate a simple model of multi-state innovation with two firms. Competition takes the form of a race. The position of each player is denoted by a scaler with player 1 at x and 2 at y. Success is defined by one player crossing 0; therefore we assuame x and y are initially both negative and that the current state of the race is represented by a point in the third quadrant of the plane. A player can attempt to improve its position by investments which determine the probability of a jump to a better state of knowledge. Jumps occur in two ways. There are *partial jumps* which, if a player is at a point $x < 0$, have a probability of $F(x)$ of hitting 0 and otherwise have a probability of $f(s, x)ds$ of landing in the interval $(s, s + ds)$, $s < 0$. There are also *leaps* from a to 0, the probability of which is proportional to both investment in that process and $G(x)$ if a player is at x. The leaps will be called more risky since whenever investment is such that leaps and partial jumps have the same expected jump, the expected gain in the value of any convex function of position is greater for leaps. For the sake of simplicity, we assume square cost functions.

The following notation summarizes the basic model:

$x\ (y) \leq 0$	State of firm 1 (2).
$udt\ (vdt)$	Probability that a partial jump of $x(y)$ occurs with $u(v)$ being chosen by firm 1 (2).
$f(s,\, x)ds$	Probability of jump from x to $(s, s+ds)$ if a partial jump occurs. If $s < x$, then $f(s,\, x) = 0$. Otherwise, we assume that the distributions of the jumps are ordered by first-order stochastic dominance, that is, if $x' > x$, then $f(s,\, x')$ first-order stochastically dominates $f(s,\, x)$. f is bounded above.
$F(x)$	Probability that a partial jump hits 0 from a if a partial jump occurs. $F(x)$ is increasing in x, by the stochastic ordering of f in x. F is positive everywhere. $F(x) = 1 - \int_{[x,\,0)} f(x,\, a)ds$.
$wG(x)dt,\ (zG(y)dt)$	Probability that firm 1 (2) leaps to 0 from $x(y)$, where firm 1 (2) chooses $w(z)$. G is bounded above and positive everywhere.
$\alpha u^2/2 + \beta w^2/2$	Firm 1's costs and the social costs associated with its choice of u and w. $\alpha,\, \beta > 0$
$\alpha v^2/2 + \beta z^2/2$	Firm 2's costs and the social costs associated with its choices of y and z.
$P > 0$	Prize to winner. There is no prize for the loser.
$B > 0$	Social benefit of success.
$\rho > 0$	The social and private discount rate.

This model differs from earlier multi-stage models in substantial ways. In the multi-stage analysis of Reinganum (1985), when one firm succeeds in achieving stage n, all firms are able to compete equally for being first to achieve stage $n+1$; therefore no firm is able to pull away from the others. Similarly, in Lee (1982) and in Telser (1982), a firm may pull away in the sense that it may achieve an increasingly superior cost structure, but the leading firm has no advantage in achieving any other low level of costs. In this model, a firm may pull away from its competition and final success is easier to achieve the more advanced it is.

The ability to pull away and attain some dynamic advantage is present in models analyzed in Fudenberg, Gilbert, Stiglitz, and Tirole (1983) and in Harris and Vickers (1985) but they both assume very special structures for innovation costs and limit the investment choices of innovators. In particular, innovation is a natural monopoly in Harris and Vickers' model in that society would only want one innovation project commanding resources, a feature which limits the ability to address issues in patent policy and the structuring of incentives for innovation. Under our assumptions, however, there is a social value to having resources allocated to each innovation project since the marginal cost of effort is zero when the effort level is zero for each project.

Both Fudenberg et al., and Harris and Vickers focus on conditions under which a firm will surely win the patent race once it has any small advantage over its competitor. The information lag model studied in Fudenberg et al. paper is closely related to our model. In both models no player knows what the other player is currently doing, but both know the position of its opponent at the beginning of each period. The models differ in that the state of each player responds stochastically to his efforts whereas Fudenberg et al. assume a deterministic response. They also make an increasing cost assumption concerning the relationship between effort and progress, but must make restrictive assumptions to render the analysis tractable.

All previous dynamic models have assumed only one kind of research investment. By permitting alternatives of varying riskiness, we can also compare the relative allocation of resources among projects of varying riskiness.[2] Finally, we also determine how the relative efficiency of the two firms is related to their relative position, finding that the lagging firm is less efficient. We address the issue of when a competition should be ended and a winner granted the monopoly right to the innovation, a question previously ignored.

We will see that this general model can be used to address several issues in the economics of innovation competition. Before analyzing our model we will first discuss our approximation approach and what it can yield.

3. Approximations

The model described above is far too general to hope for a closed-form solution, a common goal of such analyses. Nor will the structure be sufficiently tractable so as to allow for comparative static analysis as in previous work. We will instead use basic approximation techniques to study our general model for cases near some tractable case. This section reviews the basic mathematics underlying our approach and discusses its usefulness.

The primary tools used below are generalizations of Taylor's theorem and the Implicit Function theorem in R^n to Banach spaces. Taylor's theorem for a real-valued function over R^n, $f(x; z)$ (think of x as the variable and z as a parameter) says that if $f(x; z)$ is c^n in x on $[0, b]$, then for any z and any a ε $(0, b)$ there is a

[2] Dasgupta and Stiglitz (1980b) also model riskiness choice. However, their analysis is of questionable validity since their equilibrium equation, (36), often does not have a solution. In particular, it cannot have a solution if $N = 1$ and riskiness is strictly increasing in α since there is no cost to increasing α and increasing riskiness is always of value in their model. This possibly explains why their conclusions contradict those of this study. Bhattacharya and Mookherjee (1984) have also examined a static portfolio choice problem.

$c \; \varepsilon \; (0, \, a)$ such that

$$f(a;z) = \sum_{k=0}^{n-1} f^{(k)}(0;z)\frac{a^k}{k!} + f^{(n)}(c;z).$$

This states that the k-th degree polynomial in Taylor's Theorem is an $0(a^n)$ approximation of $f(a;z)$ for a to the right of 0. In particular, properties such as positivity and convexity which hold for this approximating polynomial near zero also hold for $f(x;z)$ when x is near zero.

Since equilibria in our games will be expressed as a collection of functional equations of the equilibrium strategies, we will use the Implicit Function Theorem to "compute" equilibria for games "close" to games for which solutions are known. Generally, the Implicit Function Theorem states that f can be uniquely defined for x near zero by a relation of the form $H(x, f(x); z) = 0$, wherever $H_1(0, f(0); z)$ exists and $H_2(0, f(0); z) \neq 0$. This allows us to implicitly compute the derivatives of f with respect to x as a functions of x and z, leading to a polynomial approximation for f.

However, our strategies are not going to be vecotrs of real numbers, but rather functions of the state variable, objects which are from infinite-dimensional spaces. It is necessary, therefore, to first introduce some terminology from nonlinear functional analysis. This will allow us to generalize the Implicit Function Theorem to functions and power series over Banach spaces and implement an approximation approach. Suppose that X and Y are Banach spaces, i.e., normed complete vector spaces. A map $M : X^k \to Y$ is *k-linear* if it is linear in each of its k arguments. It is a *power* map if it is symmetric and k-linear, in which case it is denoted by $Mx^k \equiv M(x, x, \ldots, x)$. The norm of M is constructed from the norms on X and Y, and is defined by

$$||M|| = \sup_{||x_i||=1, \; i=1,2,\ldots,k} ||M(x_1, x_2, \ldots, x_k)||$$

For any fixed x_0 in X, consider the infinite sum in Y:

$$Tx = \sum_{k=1}^{\infty} M_k(x - x_0)^k$$

where each of the M_k is a k-linear power map from X to Y. When the infinite series converges, T is a map from X to Y. It will be convenient to associate a real valued series, called its *majorant series*, with T

$$\sum_{k=0}^{\infty} ||M_k|| \, ||x - x_0||^k$$

The important connection between the power series for T and its majorant series is that T will converge whenever its majorant series does.

Definition: T is analytic at x_0 if and only if it is defined for some neighborhood of x_0 and its majorant series converges for some neighborhood of x_0.

With these definitions, we can now state the analytic operator version of the Implicit Function Theorem.

Theorem 1. *Implicit Function Theorem for Analytic Operators: Suppose that*

$$F(\varepsilon, x) = \sum_{n,k=0}^{\infty} \varepsilon^n M_{nk}\, x^k \tag{1}$$

defines an analytic operator, $F : U(0,0) \subset X \to Y$, where $U(0,0)$ is a neighborhood of $(0,0)$ in RxX. Furthermore, assume that $F(0,0) = 0$ and that the operator $M_{01} : X \to Y$, representing the Frechet cross-partial with respect to x at $(0,0)$, is invertible. Consider the equation

$$F\left(\varepsilon, x(\varepsilon)\right) = 0 \tag{2}$$

implicitly defining a function $x(\varepsilon) : R \to X$. The following are true:

1. There is a neighborhood of $0\ \varepsilon R$, $V(0)$, and a number, $r > 0$, such that (A2) has a unique solution of $||x|| < r$ for each ε in $V(0)$.

2. The solution, $x(\varepsilon)$, of (A2) is analytic at $\varepsilon = 0$, and, for some sequence of x_n in X, can be expressed as
$$x(\varepsilon) = \sum_{n=1}^{\infty} x_n\, \varepsilon^n \tag{3}$$
where the coefficients x_n can be determined by substituting $(A3)$ into $(A1)$ and equating coefficients of like powers of ε.

3. The radius of convergence of the power series representation in (ii) is no less than that of the analytic map, $z(\varepsilon) : R \to R$, defined implicitly for some neighborhood of 0 by
$$0 = \sum_{n,k=0}^{\infty} \varepsilon^n\ ||M_{nk}||\ z(\varepsilon)^k \tag{4}$$
Furthermore, for some sequence z_n of real numbers,
$$z(\varepsilon) = \sum_{n,k=0}^{\infty} \varepsilon^n\ z_n$$
represents the solution to $(A4)$ and $|z_n| > ||x_n||$.

See Zeidler (1986).

As will be apparent, the mathematics turns out to be elementary since our task is reduced to recursive computation of x_n terms. The term-by-term approach alluded to in (ii) above will be illustrated in the next section.

However, we should first discuss the value of such an approximation approach. Our objective below is to apply it to examine subgame-perfect equilibria in our model. In most of the analyis below, we will express equilibrium strategies and values as functions of the prize, P, social benefit, B, and the position, (x, y) and examine approximations for them around the case of a zero prize and no social value. At first blush, approximations based on such cases may appear useless since the case of a zero prize degenerate. A number of considerations justify the effort and indicate the general value of this approach.

First, the approximations can provide counterexamples to conjectures. Suppose $g_1(P)$ and $g_2(P)$ are functions of interest, and it is initially conjectured that $g_1(P) > g_2(P)$. If we can show that $g_1(0) = g_2(0)$ and $g_1(0) < g_2'(0)$, then there must be an interval of $P > 0$ where $g_1 > g_2$, contradicting the conjecture. This in fact will occur below when we discuss equilibrium reaction functions.

Furthermore, suppose g_1 depended on some function F, i.e., $g_1(P; F)$. More generally, one could identify conditions on F which lead to the "$g_1 > g_2$" conjecture failing. In models of dynamic competition, we often make special assumptions about the functional form of such F's. After deriving our results, however, we usually don't know *exactly* what general feature of the functional form was crucial. Our approach below will find exactly what features of all structural elements are critical for any results for the case of a small prize. Whenever the intuition gathered from such an analysis does not depend on P being nearly 0, then we have perhaps discovered a robust feature of the model. Generally, we study such approximations not because they are valid for nearly degenerate cases, but rather that they likely indicate patterns which continue to hold much more generally.

Second, *any* analytical investigation of this model must focus on cases which are degenerate in some ways. Note that the models of Lee and Wilde, Reinganum, and Fudenberg et al. are all special cases of this general model (or some slightly different general model) which are degenerate in some dimension. For example, Lee and Wilde, and Reinganum implicitly assume that the success probability function $G(x)$ is independent of the position x, making position irrelevant. Also, $F(x)$ is essentially absent in their models, as if α were infinite. Each of these special cases are of interest despite their degeneracies. However, if we are interested, for example, in a precise look at how innovators react to each other's successes, it is valuable to look at cases in which there are as few unmotivated restrictions on the underlying stochastic structure as possible. It is unfortunate that we may have to assume a small prize, but that is the price we pay here to attain this particular goal. Finally, the technique that is exploited below can be used generally to develop a robustness analysis for all the special cases studied previously.

The main advantage of our approach is that one can examine the model one wants to study, not search for special solvable cases. In general, the only ingredient needed is some case with a tractable and usable solution, which will provide a basis for a more general analysis. That is the value of perturbation analysis in the physical sciences. For example, the Einstein equations of general relativity theory are generally intractable. However, many of that theory's powerful implications, such as gravatational radiation, have come from the examination of the high-order approximations of solutions to the field equations around the case of no matter. Low-order approximations have often been useful in economics; in macroeconomics, we often use linearizations of dynamic systems around their steady states and in public finance, we often use the rule-of-thumb that approximates the excess burden of a tax with the product of the demand elasticity and the square of the tax rate. The usefulness of high-order approximations in our context will be apparent below.

4. An Example: The Case of a Single Firm

In this section we will analyze the case of a single firm. This will illustrate the analysis used below and will also be used later when we examine the optimal stage at which to end the race. Also, to cut down on inessential clutter, we will examine here only the simple case when β is infinite. The general solution will be displayed at a later point.

The case of a single innovator is a dynamic programming problem. If $M(x)$ is the value of position x to the firm, that is, the supremum of the expected present values of payoffs under all possible strategies for a firm currently at x, then the dynamic programming equation for M is

$$M(x) = \max_u \Big\{ -\frac{\alpha u^2}{2}dt + M(x)(1-\rho dt)\ (1-udt) \qquad (5)$$
$$+(1-\rho dt)udt\left(\int_x^0 M(s)f(s,x)ds\right) + udtPF(x)\Big\}$$

where dt is the infinitesimal unit of time.[3] The individual terms of the maximand represent the expected value of innovative effort. If the rate of effort is u, the expenditure during dt is $-(1/2)\alpha u^2 dt$. With probability $1-udt$ there will be no success, implying that the state of knowledge dt units of time in the future will remain x and the value will remain $M(x)$. The current unconditional expected value of that event is $(1-\rho dt)(1-udt)M(x)$. With probability udt there will be a jump to some $s\ \varepsilon\ (x,0]$. If x jumps to 0, an event with probability $F(x)$ conditional on a jump occurring, the

[3] Throughout this essay we will employ the intuitive infinitesimal notation of equation (1). However, all the dynamic programming equations can be derived formally, as in Bryson and Ho.

immediate reward is P. Since the reward is immediate, no discounting occurs. If x jumps to a point $x' \varepsilon\ (s,\ s+ds)$, an event with a conditional probability of $f(s,x)ds$, the value becomes $M(s)$ in the next period. In the foregoing, $\int_x^0 \ldots\ ds$ will represent $\int_{[x,0)} \ldots\ ds$, thereby ignoring the atom at $x = 0$. We use this notation to distinguish reaching an intermediate stage from that of winning. (1) therefore states that the value of a position equals the maximum expected current value of future positions net of current costs. This is just the principle of optimality of dynamic programming.

Solving the maximization problem in (1) shows that

$$\alpha u = \int_x^0 M(s)f(s,x)ds + PF(x) - M(x) \tag{6}$$

Substituting this first-order condition into the control equation yields the Bellman equation for this control problem:

$$0 = \left(\int_x^0 M(s)f(s,x)ds + PF(x) - M(x)\right)^2 /2\alpha - \rho M(x) \tag{7}$$

By standard dynamic opitmization methods, there exists a unique such M.

We cannot generally find a closed-form solution for M in (3). We will instead use an approximation to give us precise information about M for any F and an open set of parameter choices. Note that this fits our discussion above. If we assume that the value function M is in the Banach space of real-valued functions on the negative reals with the supremum norm, then the RHS of (3) is the sum of a linear a bilinear operator acting on M and the real parameter P. To proceed in this fashion one should examine dimensionless versions of a problem since the concept of "small" should not depend on the choice of units. Define $m \equiv M/P$ to be the value of problem (1) relative to the prize. m is a dimensionless quantity representing the value of the game which will yield a substantive concept of small.

Rewritten in terms of m, (7) becomes the equation

$$m(x) = p\left(\int_x^0 m(s)f(s,x)ds + F(x) - m(x)\right)^2 \tag{8}$$

where $p \equiv P/2\alpha\rho$ is the size of the prize relative to the marginal cost of innovation and the cost of capital. Since the dimension of ρ is (time)$^{-1}$ and that of α is (dollars) (time), p is dimensionless and will be our measure of the prize. Since $m,\ p,\ f$, and F are all dimensionless, (3′) is a dimensionless representation of (3). When p is zero, (3′) yields the obvious solution, $m(x) = 0$. p may be zero either because P is zero or because $\alpha\rho$, the "costs", are infinite. Focussing on p makes clear that we are not assuming that the prize itself is small but rather it is small compared to the rate of increase in marginal cost. This will imply that the prize is to the first order equal to the costs and that the net profits of an innovator are small relative to the prize. The interpretation that the prize just covers the opportunity costs of innovative activity makes our focus on small p more plausible.

Once we transform (7) into a dimensionless equation, we also must transform other variables of interest; in particular, the control variable, u. However, u is not dimensionless since it measures effort per unit of time and depends on the time unit. We can rewrite (6) into the dimensionless form:

$$\tilde{u} \equiv \frac{u}{\rho} = 2p\left(\int_x^0 m(s)f(s,x)ds + F(x) - m(x)\right) \tag{9}$$

where $\tilde{u}$ is the dimensionless rate of effort per normalized unit of time.

We now illustrate computing a local solution to (3′). If $p = 0$, then $m = 0$. Applying the Implicit Function Theorem tells us that $m(x;p)$ is smooth in p for p near zero, and that we can approximate $m(x;p)$ for such p up to $0(p^n)$

$$m(x;p) \approx m(x;0) + pk^1(x) + p^2k^2(x) + \ldots + p^nk^n(x) \tag{10}$$

where we define $k^n(x) \equiv \frac{1}{n!}\frac{\partial^n m}{\partial p^n}(x,0)$. First note that $m(x;0) = 0$ since a zero prize makes the optimal value of the problem zero.

Differentiating (3′) with respect to p and evaluating at $p = 0$ shows that

$$k^1(x) = F(x)^2 \tag{11}$$

Taking a second derivative of (3′) with respect to p, evaluating it at $p = 0$, and using the fact that $\partial m/\partial p\ (x;0) = k^1(x) = F(x)^2$, we find that[4]

$$k^2(x) = 2F(x)\left(\int_x^0 F(s)^2 f(s,x)ds - F(x)^2\right) \tag{12}$$

Continuing in this fashion, one can recursively compute $k^n(x)$ for any n justified by the known smoothness of m in terms of p. Note that *no* smoothness of m in x need be assumed.

It is usually quite tedious to do all the differentiation explicitly. A standard trick in perturbation analysis is to take the polynomial approximation for m in terms of p in (4), insert it into (8), and conduct the algebraic operations indicated in (3′) to get an approximate polynomial representation of (8). (8) then becomes

$$pk^1(x) + p^2k^2(x) + \ldots = pF(x)^2 + 2p^2\left(\int_x^0 k^1(s)f(s,x)ds) - k^1(x)) + \ldots\right) \tag{13}$$

[4]As is becoming apparent, our notation will be burdened with many superscripts. Superscripts to functional names, as in $k^2(x)$, will represent distinct functions, and will *never* represent iteration as in $k(k(x))$. Superscripts to functional evaluations represent powers. Hence, $k^2(x)^3$ is the cube of the value of the function k^2 evaluated at x.

If we equate terms linear in p in (13), we find that $k^1(x) = F(x)^2$. Combining p^2 terms and using the computed solution for k^1 demonstrates (12). Continuing in this fashion will yield all k^n functions. Since this approach yields the terms of the Taylor series more efficiently, we will use it below.

From these expressions we may infer several obvious properties of the optimal control for small p. For example, that if p is small, effort increases as one is closer to the finish. This follows from the observation that the $pF(x)$ term dominates in (9) since m is $0(p)$ implying that u rises as $F(x)$, and hence x, rises. Also, u falls and as α and ρ rise, an intuitive result since both represent costs. Using this approach, we next examine the total social optimum when we have two separate projects and two firms.

4. The Social Optimum

Let $W(x, y)$ be the social value function when current states are x and y. Then the Bellman equation becomes

$$\begin{aligned} & W(x,y) = \max_{u,v,w,z}(-\alpha u^2/2 - \alpha v^2/2 - \beta w^2/2 - \beta z^2/2)dt \\ & +udt\left(\textstyle\int_x^0 W(s,y)f(s,x)ds + BF(x)\right)(1-\rho dt) \\ & +vdt\left(\textstyle\int_y^0 W(x,s)f(s,y)ds + BF(y)\right)(1-\rho dt) \\ & +\left(wG(x)+zG(y)\right)(1-\rho dt)Bdt \\ & +(1-\rho dt)\left(1-(u+v+wG(x)+zG(y))dt\right)W(x,y) \end{aligned} \tag{14}$$

(14) is derived just as (5) was. The first-order conditions of (14) imply

$$\begin{aligned} & \alpha u = \textstyle\int_x^0 W(s,y)f(s,x)ds + BF - W(x,y) \\ & \beta w = G(x)\left(B - W(x,y)\right) \end{aligned}$$

αv and βz may be expressed similarly. Using the first-order conditions, (**??**), for u and w, the corresponding conditions for v and z, (14) becomes

$$\begin{aligned} 0 \; = \; & \left(E_x\left\{W(s,y)\right\} - W(x,y)\right)^2/2\alpha + \left(E_y\left\{W(x,s)\right\} - W(x,y)\right)^2/2\alpha \\ & + \left(G(x)\left(B - W(x,y)\right)\right)^2/2\beta + \left(G(y)\left(B - W(x,y)\right)\right)^2/2\beta - \rho W(x,y) \end{aligned} \tag{15}$$

where

$$E_x\left\{W(s,y)\right\} \equiv \int_x^0 W(s,y)f(s,x)ds + BF(x)$$

and

$$E_Y\left\{W(x,s)\right\} \equiv \int_y^0 W(x,s)f(s,y)ds + BF(y)$$

Theorem 2. *There exists a unique solution, $W(x, y)$, to the social optimum problem, and $W(x, y)$ is analytic in B, α, β, and ρ.*

Proof: The RHS of (15) is an analytic operator on bounded functions over the nonpositive reals. When $P - 0$, the unique solution is $W = 0$. Furthermore, the cross Frechet derivative, first with respect to P then with respect to W, is $-\rho$, which is an invertible operator on bounded functions. Therefore, we can invoke the Implicit Function Theorem to assert Theorem 1.

We next compute an approximation for W. Suppose $W(x,y) = B(bh^1(x,y) + b^2h^2(x,y) + \ldots)$ is the approximating series for W around $B = 0$, which exists by the Implicit Function Theorem. We let $b = B/2\alpha\rho$ be a dimensionless measure of the social value, and use it since the implied representation for W/B will be dimensionless. The linear term, h^1, is computed to be

$$h^1(x,y) = F(x)^2 + F(y)^2 + \gamma\left(G(x)^2 + G(y)^2\right) \tag{16}$$

and the investment rules are approximated to $0(b^2)$ by

$$\begin{aligned} \tfrac{u}{\rho} &\approx 2bF(x) + 2b^2\left(\textstyle\int_x^0 h^1(s,y)f(s,x)ds - h^1(x,y)\right) \\ \tfrac{w}{\rho} &\approx 2\left(b - b^2h^1(x,y)\right)\gamma G(x) \end{aligned} \tag{17}$$

and similarly for v and z. The first-order approximations for u and w are as if the current hazard rate of immediate success was common to all stages since $\alpha u \approx BF(x)$ and $\beta w \approx BG(x)$ to $0(B)$. This indicates that the first-order behavior of this multi-stage game at any stage reduces to the behavior of a single-stage game. In particular, to a first order, the presence of other projects has no impact on investment rules. Intuitively, this is because for small B, effort levels are "small," the probability of success for any one project is "small," and by independence the probability of success by two projects is "small squared," hence negligible. Therefore, most of the interesting multi-stage questions will require examination of h^1 and h^2 which appears in the $o(B)$ terms. We will return to this in Section 8.

Straightforward combinations of (**??**) and (17) prove Corollary 1.

Corollary 3. : *For small B, the following hold for the optimal innovation policy:*

(i): as $x(y)$ increase, $u(v)$ and $w(z)$ increase and $v(u)$ and $z(w)$ fall;
(ii): $w(z)$ is increasing and concave in B;
(iii): $u(v)$ is increasing in B but may be convex or concave in B;
(iv): W is increasing and convex in (x,y) if $F(x)$ and $G(y)$ are convex;
(v): u and v (w and z) are decreasing in ρ and $\alpha(\beta)$; and
(vi): w and z are decreasing in α.

Particularly note that, if the two firms were managed in a socially optimal fashion, each firm would increase its efforts on both projects as it advances, and the other would decrease its effort. Also, the magnitude of these reactions are on the order of B^2. These features will be substantially different in the equilibrium of the R&D race.

5. Equilibrium of the Innovation Game

We next solve for the symmetric subgame-perfect equilibrium of the corresponding game. We are implicitly assuming that the current states of both players are common knowledge since if we had assumed that no player could observe the position of his competitor then the open-loop solution would be the correct equilibrium concept. While this common knowledge aspect is certainly valid in sports races, it may appear awkard here. It asserts that player 1's knowledge of the value of y has no impact on the value of x, i.e., that a firm may know how much its opponent knows without knowing exactly what its opponent knows. This is not an unrealistic description of matters in knowledge-intensive activities. Academics, for example, should not be uncomfortable with this assumption since they often judge colleagues' relative levels of knowledge about a subject without having an equivalent level of expertise in the area. In sum, we are assuming that firms may determine their relative positions without actually having access to each other's knowledge. It will also be sometimes true that players will want to reveal their position if they can do so without revealing useful knowledge. For these reasons, we stay with the race analogy.

Let $V(x,y)$ represent the value to firm 1 of state (x,y). We will examine symmetric equilibria, implying that $V(y,x)$ will represent the value to firm 2 of state (x,y). We also limit our examination to equilibria which depend only on the current state of the game.[5] The Bellman equation for firm 1 is

$$\begin{aligned} V(x,y) \;=\; & \max_{u,w} \Big\{ -\left(\alpha u^2/2 + \beta w^2/2\right) dt + wG(x)dtP(1-\rho dt) \\ & + udt \left(\int_x^0 V(s,y) f(s,x) ds + PF(x) \right) (1-\rho dt) \\ & + vdt \left(\int_y^0 V(x,s) f(s,y) ds \right) (1-\rho dt) \\ & + (1-\rho dt)\left(1-(u+v+wG(x)+zG(y))dt\right) V(x,y) \Big\} \end{aligned} \tag{18}$$

The first-order conditions from (18) allow us to express its strategy in terms of

[5] Implicitly, we are ruling out reputation effects, trigger strategies, and other phenomena which can support implicit collusion in such infinite-horizon dynamic games. This is reasonable in the case of leap investment since such investments are unobserved and any cheating could be inferred only when a leap occurred, which would be too late. Some implicit collusion in partial jump investment is probably possible since, as long as neither had won, each could infer cheating if the other seemed to be moving too quickly. However, any such implicit collusion would be imperfect since one could never exactly distinguish between cheating and good luck. Finally, note that we are computing the unique limit, as the horizon increases to infinity, of the finite-horizon equilibria since the latter are unique in the cases we study.

the value function at that point and later points:

$$\begin{aligned} \alpha u(x,y) = &\quad \textstyle\int_x^0 V(s,y) f(s,x) ds_P F(x) - V(x,y) \\ \beta w(x,y) = &\quad (P - V(x,y))\, G(x) \end{aligned} \tag{19}$$

$$\alpha u(x,y) = \int_x^0 V(s,y) f(s,x) ds_P F(x) - V(x,y) \qquad (13a)$$

$$\beta w(x,y) = (P - V(x,y))\, G(x) \qquad (13b)$$

By symmetry, the strategies of firm 2 are

$$\begin{aligned} \alpha v(x,y) = &\quad \textstyle\int_y^0 V(s,x) f(s,y) ds + PF(y) - V(y,x) \\ \beta z(x,y) = &\quad (P - V(y,x))\, G(y) \end{aligned} \tag{20}$$

$$\alpha v(x,y) = \int_y^0 V(s,x) f(s,y) ds + PF(y) - V(y,x) \qquad (14a)$$

$$\beta z(x,y) = (P - V(y,x))\, G(y) \qquad (14b)$$

The characterization equation for equilibrium is found by substituting these equations for strategies into the Bellman equation, which then reduces to

$$\begin{aligned} 0 \;=\; & \left(\int_x^0 V(s,y) f(s,x) ds + PF(x) - V(x,y) \right)^2 /2\alpha + (P - V(x,y))^2\, G(x)^2/2\beta \\ & + \left(\int_y^0 V(s,x) f(s,y) ds + PF(y) - V(y,x) \right) \left(\int_y^0 V(x,s) f(s,y) ds - V(x,y) \right) /\alpha \\ & - \left(\rho + \frac{(P - V(y,x)) G(y)^2}{\beta} \right) V(x,y) \end{aligned}$$

Theorem 4. *There exists a $\bar{P} > 0$ such that for $P \varepsilon [0, \bar{P}]$, there is a symmetric subgame perfect equilibrium $V(x,y)$, which is analytic in P, α, β, and ρ, and represented as a solution to (15).*

Proof. Same as Theorem 1.

Suppose $V(x,y) = P(pg^1(x,y) + p^2 g^2(x,y) + \ldots)$ is a Taylor series approximation of $V(x,y)$ for small p. By Theorem 2, such a representation exists and is unique for small p.[6] By substituting this representation for V in (15) and equating coefficients

[6] Even though (**??**) is not expressed in p, it can be straightforwardly rewritten so that V/P, the dimensionless value of the game, depends on P, α, β, and ρ only through p and the dimensionless ratio α/β.

of like powers, we find

$$
\begin{aligned}
g^1(x,y) &= F(x)^2 + \gamma G(x)^2 \\
g^2(x,y) &= 2F(x)\left(\int_x^0 (F(s)^2 + \gamma G(s)^2) f(s,x) ds - F(x)^2 - \gamma G(x)^2\right) \\
&\quad -2\left(\gamma G(x)^2 + \gamma G(y)^2 + F(y)^2\right)\left(F(x)^2 + \gamma G(x)^2\right)
\end{aligned} \tag{21}
$$

The equilibrium strategies are therefore approximated to $0(p^3)$ by

$$
\begin{aligned}
\frac{u(x,y)}{\rho} &\approx 2pF(x) + 2p^2\left(\int_x^0 g^1(s,y) f(s,x) ds - g^1(x,y)\right) \\
&\quad +2p^3\left(\int_x^0 g^2(s,y) f(s,x) ds - g^2(x,y))\right) \\
\frac{w(x,y)}{\rho} &\approx 2\gamma p\left(1 - pg^1(x,y) - p^2 g^2(x,y)\right) G(x)
\end{aligned} \tag{22}
$$

and similarly for $v(x,y)$ and $z(x,y)$. This solution and its approximation now allows us to compare equilibrium with the social optimum and evaluate the competitive equilibrium allocation of resources.

6. Comparisons of the Optimal and Equilibrium Outcomes

We next will compare the levels of innovative activity under social control with those levels in the game equilibrium. If $P = B$, the difference between innovative effort under competition, u^c, w^c, and the socially optimal levels, u^s, w^s, is expressed, up to $0(p^2)$, by

$$\rho^{-1}(u^s - u^c) \approx -2p^2\left(F(y)^2 + \gamma G(y)^2\right) F(x) \tag{23}$$

$$\rho^{-1}(w^s - w^c) \approx -2\gamma p^2\left(F(y)^2 + \gamma G(y)^2 + \gamma G(y)^2\right) G(x) \tag{24}$$

The difference between firm two's choices, v^c, z^c, and the optimal controls v^s, z^s, are similarly expressed. First note that there is excessive investment in all projects under competition, a conclusion common in these models. The excess is greater as either firm is closer to success. Also the excess investment relative to the socially optimal investment increases for each firm as the other firm is closer to success. These results are expected since each firm ignores the social value of the other's presence in the innovation process (see Mortenson (1982)).

We also note that it is not clear which firm is more excessive in R&D investment. If E_{uv} is the difference, $(u^c - u^s) - (v^c - v^s)$, between the two competitor's excessive investment in their partial jump processes, then

$$E_{uv} \approx 2\gamma\rho p^2\left(F(y)F(x)F(y) - F(x)\right) + G(y)^2 F(x) - G(x)^2 F(y)$$

to $0(p^2)$. If there are no "leaps", $G \equiv 0$ and then $E_{uw} < 0$ if $x > y$, that is, the laggard's investment is more excessive than the leader's. This holds also if the leap and partial jump processes are sufficiently similar, in particular if $G = \lambda F$ for some scaler $\lambda > 0$. However, if $F(y)$ is small but $G(y)$ is not, then $E_{uw} > 0$, and the leader invests more excessively in partial jumps.

In relative terms, however, we can be more precise since

$$\frac{(u^c - u^s)}{u^s} \approx p\left(F(y)^2 + \gamma G(y)^2\right) \tag{25}$$

is increasing in y. $(w^c - w^s)/w^s$ is similarly found to be increasing in y. The dependence of $v^c - v^x$ and $z^c - z^s$ on x are symmetrically expressed. Therefore, the laggard's excess investment in both partial jumps and leaps expressed as a fraction of the socially optimal investment is greater. Theorem 3 summarizes these comparisons.

Theorem 5. *If B is small and $P = B$ then*

$$\frac{u^c - u^s}{u^s} > \frac{v^c - v^s}{v^s} \quad \text{and} \quad \frac{w^c - w^s}{w^s} > \frac{s^c - z^s}{z^s}$$

if and only if $x < y$.

These comparisons do not necessarily say anything about the efficiency of resource allocation given that there is competition. For example, in deciding whether to subsidize the current leader a social planner should consider its impact on the future nature of the distorted allocation of resources due to the competition. We next address this issue for the case $P = B$.

If $P = B$, the social value of the game is $V(y, x) + V(x, y)$ since all benefits of innovation are appropriated by the firms. At any position, the net social marginal values, $NSMV$, of u and w per dollar of expenditure equal the ratio of the net contribution to the social value and the marginal cost:

$$\begin{aligned} NMSV_u &= \frac{\int_x^0 V(y,s)f(s,x)ds - V(y,x)}{\int_x^0 V(s,y)f(s,x)ds + PF(x) - V(x,y)} \\ NSMV_w &= -\frac{V(y,x)}{P - V(x,y)} \end{aligned} \tag{26}$$

where we use (19) to simplify expressions. Using our expansion for $V(x, y)$, (26) implies that, as p converges to 0,

$$\begin{aligned} p^{-1}NMSV_u &\approx -g^1(y,x) = -F(y)^2 - \gamma G(y)^2 \\ p^{-1}NMSV_w &\approx -F(y)^2 \end{aligned} \tag{27}$$

Symmetric expressions for $NMSV_v$ and $NMSV_z$ hold. If $x > y$ then $F(x) > F(y)$ and $G(x) > G(y)$, implying that $NMSV_z$, and $NMSV_v < NMSV_u$. Therefore, the social value of more investment in either project is greater at the leading firm, even when we consider the distortions implicit in the competition.

Theorem 6. *If $P = B$ and P is small, social welfare at any stage would be increased by shifting innovation effort from the laggard to the leader. That is, if (x, y) is the current state and $x > y$, $V(x, y) + V(y, x)$ is increased if $u(x, y)^2$ is increased and $w(x, y)^2$ is decreased by ε, for small $\varepsilon > 0$, and similarly for $z(x, y)$ and $w(x, y)$.*

Theorem 4 shows that any small temporary subsidy/tax scheme which reallocates effort towards the leader is socially desirable since combinations of subsidies and taxes can induce such a switch and the objective of $V(x, y) + V(y, x)$ ignore any redistributive component of such a policy. Therefore, in this limited sense, policy should favor the current leader over the laggard.

Another interesting issue which we can address in this model is that of the efficiency of the allocation of resources between the risky leaps and the less risky partial jumps. The social efficiency of the portfolio choice by firm one is determined by comparing the net social marginal values of u and v. $NSMV_u > NSMV_w$ iff $g^1(x, y) - \int_x^0 g^1(s, y)f(s, x)ds < F(x)g^1(x, y)$ which is true since $g^1(x, y)$ is increasing in x. Hence, there is an excessive share of resources allocated to the "risky" project. To get an intuitive grasp on this result, we should compare the social valuation of the intermediate stages with the equilibrium valuation by firm one. Since the difference between g^1 and h^1 is independent of x, we need to compare g^2 with h^2 to study differences relevant for one's portfolio choice between u and w. Straightforward manipulation of the expansions for V and W shows that, ignoring terms which are of $o(P^3)$,

$$V(x, y) - W(x, y) \approx 2p^2 \left(F(x)^2 + \gamma G(x)^2\right) \left(F(y)^2 + \gamma G(y)^2\right) P + Z(y) \qquad (28)$$

where $Z(y)$ depends only on y. Therefore, $V - W$ is increasing in x for small p. First, this implies that investment is even more excessive than indicated by p^2 terms since the gap between social and private values of R&D is increasing at $0(P^3)$. Second, it indicates a bias towards risky R&D projects. Since this excess increases in x, those projects which are more likely to yield big jumps, holding the expected jump constant, will find their private value to be more excessive relative to their social value.

Theorem 7. *If $P = B$ and P is small, social welfare would be increased if resources were shifted from the risky R&D projects to the less risky projects.*

The last comparison we will make is between the optimal and equilibrium reactions of firms to eacha other's partial successes. Before using our approximations, note that our expression for firm 1's equilibrium choice of w (19), differs substantially from the expression for the social choice, (**??**), despite their formal similarity. In (**??**), it is clear that the optimal choice of w falls if the social value of the social position (x, y) increases but x, the position of firm 1, remains unchanged. In particular, an advance in firm two's position will increase the social value, and hence lead to a reduction of expenditure at firm one on the leap investments. In (19), we find that expenditure on w will rise as the value of the game to firm one falls, which is the expected response to an advance by firm two. Hence, if the social and private value functions vary with position in the intuitive fashion, firm one will increase leap investments in response to an advance by firm two, even though the socially optimal response would be a reduction in effort.

Proving these conjectures globally would be quite difficult given the nonlinear nature of the expression for the equilibrium value functions. However, our approximations will immediately confirm them. Since $g^2(x, y)$ is independent of y, the dependence of u and w on y for small P, is determined by the dependence of g^3 on y, and is summarized in

$$\begin{aligned} \rho^{-1}u^c = & \quad \ldots + 2p^3 \left(F(y)^2 + \gamma G(y)^2\right) \\ & \qquad \times \left(\int_x^0 (F(s)^2 + \gamma G(s)^2) F(s,x) ds - F(x)^2 - \gamma G(x)^2 \right) \\ \rho^{-1}w^c = & \quad \ldots + 2\gamma p^3 \left(F(y)^2 + \gamma G(y)^2\right) \ \left(F(x)^2 + \gamma G(x)^2\right) \ G(x) \end{aligned} \tag{29}$$

where we have displayed all terms of $0(P^3)$ which depend on y.

Theorem 8. *If $P = B$ and P is small,*

$$0 < \left| \frac{\partial U^c}{\partial y} \right| < -\frac{\partial u^s}{\partial y}\,, \ \frac{\partial w^c}{\partial y} \ > 0 > \frac{\partial w^s}{\partial y}\,,$$

that is, one's equilibrium reactions are less than the optimal reactions in magnitude. Furthermore, $\partial u^c/\partial y$ is always positive and $\partial w^c/\partial y$ is of ambiguous sign. Symmetric results for firm two hold.

Proof: The comparisons of magnitude follow from the fact that $\partial u^c/\partial y$ is $0(p^3)$ by (29) but $\partial u^s/\partial y$ and $\partial w^s/\partial y$ are $0(p^2)$ by (11a). The sign conditions for w^c and z^c follow from (29b). If $F(s)$ and $G(s)$ are large relative to $F(x)$ and $G(x)$ for $s > x$, then the integral in (29a) dominates and $\partial u^c/\partial y > 0$. However, if $F(s) \approx F(x)$ and $G(x) \approx G(x)$ for $s > x$, then $\int_x^0 (F(s)^2 + \gamma G(x)^2) f(s,x) ds \approx (F(x)^2 + \gamma G(x)^2)\ (1 - F(x))$ and $\partial u^c/\partial y < 0$ in (29). **Q.E.D.**

In comparing the dependence of strategies on the positions of the players, first note that there is no reaction of one firm to another's position to $0(p^2)$. Hence, the equilibrium reactions of the firms to each are smaller than the optimal reactions. Furthermore the direction may be wrong. In the case of leap investment, the reaction will always be in the wrong direction. This is intutitively seen from (19): we expect that as firm two advances, the value of the game to firm one, $V(x, y)$ decreases, thereby raising one's choice of w. In the social control case, the value increases as firm two advances, reducing the social choice for w.

However, the reaction of u is ambiguous. The reaction of a partial jump's control to the other firm's movement depends on just how different the stages are. If the stages are similar in that the probability of winning immediately per unit of effort with a leap, $G(x)$, or partial jump, $F(x)$, is nearly as large at x as at any later stage, then u will fall, whereas if later stages have substantially greater likelihoods of getting one to success, then a firm's effort in partial jumps may increase as its opponent moves ahead. In the latter case, the improvement in the opponent's prospects prompts one to work harder, as if one must either work hard or concede the race. Also note that if a mean preserving spread in the probability weights $f(s,x)$ will increase the likelihood of a perverse reaction for u since the integral in ($29b$) has a convex integrand.

Finally note that firm one's choice of its leap control reacts more to an opponent's improvement as firm one is closer to final success. This indicates that effort levels are more volatile as the game is nearing completion.

At this point we should expand on the appropriate interpretation of our juggling of these various orders of magnitude. For example, the fact that the reaction of u^c to y is zero at $0(p^2)$ and possibly nonzero only at $0(p^3)$ does not imply that reactions are generally unimportant and uninteresting when compared to the effects which show up at $0(p^2)$. In fact, in many games where reactions are generally important we would find that, as the payoffs go to zero, the reactions go to zero faster than other elements of equilibrium strategies. Only for nearly degenerate games does the order reflect the relative importance of various effects. Since our objective is to gain more general insight, we make no comparisons. On the other hand, one cannot infer that an $0(p^3)$ effect will eventually dominate any $0(p^2)$ effect since other, even higher, orders also contribute. Our goal in these calculations is to sign various effects and determine the critical structural elements for an open set of games, hoping to elicit general qualitative insights about the nature of the subgame equilibria. Arguments which mix various orders of magnitudes are either illegitimate or focus too tightly on the small p nature of the analysis.

7. Implications for Social Innovation Policy

We next examine the optimal values of two parameters of social innovation policy, the portion of social benefits to be awarded to the winner and the stage at which a patent is to be granted, in this two firm innovation game. We will find that when B is small, the difference between the optimal P and B is negligible relative to B. This result validates our focus on the case $P = B$ in the previous section since it implies that all those results continue to hold for an optimally chosen P. In particular, this shows that the misallocation of resources between projects of varying riskiness will not change with an optimally chosen P. While these results are not surprising, it is instructive to show how to rigorously demonstrate them within our approach.

Let $P - \theta B$, i.e., θ is the portion of social benefits of innovation which the innovator is allowed to appropriate. We are making the simplifying assumption that this allocation of social benefits to the innovator can be made in a nondistortionary fashion. In the case of patents this is only valid if demand is inelastic. If a prize is awarded, this assumes that it is financed by nondistortionary revenue sources.

Presumably, θ is a parameter at least partially chosen by policy markers. Given that we found that there was excessive allocation of resources for innovation in the equilibrium of the innovation game, the optiaml θ is never unity. Let W again represent the social value function. Then

$$\begin{array}{rl} W(x,y) = & -\left[\alpha(u^2+v^2)+\beta(w^2+z^2)\right]\frac{1}{2}dt \\ & +(1-\rho dt)\ \left(uF(x)+vF(y)+wG(x)+zG(y)\right)Bdt \\ & +(1-\rho dt)\ \left(1-(u+v+wG(x)+zG(y))dt\right)\ W(x,y) \\ & +(1-\rho dt)\ \left(u\int_x^0 W(z,y)f(z,x)+v\int_y^0 W(x,s)f(s,y)ds\right)\ dt \end{array} \tag{30}$$

Where $u,\ v$, and z are the equilibrium policy functions if the prize is θB.

We can use the characterization in (30) to generate some information about the optimal $\theta,\ \theta^*(B)$, when B is small. This is not a completely trival calculation since any θ is optimal when $B=0$. Therefore we compute $\theta^*(0^+)$, the limit of θ^* as B falls to zero.

First, for sufficiently small $B,\ \theta^*(B)$ is defined and shown to be continuously differentiable by the Implicit Function Theorem applied to the equation $W_\theta(\theta^*(B),B)=0$, since $W_{\theta\theta}$ is not zero and $W_{\theta B}$ exists for B close to zero. Since $\theta^*(B)$ is optimal for the initial position (x,y),

$$\lim_{B\to 0^+}\ \frac{W(x,y,\theta^*(0^+),B)-W(x,y,\theta,B)}{B^2}>0$$

for all θ. Since $W(\theta,B)$ and $W_B(\theta,B)$ both converge to 0 as B converges to 0, by l'Hospital's rule this limit equals

$$\lim_{B\to 0^+}\ \frac{W_{BB}(x,y,\theta^*(0^+),B)-W_{BB}(x,y,\theta,B)}{2}$$

Therefore, $W_{BB}(x,y,\theta^*(0^+),0)-W_{BB}(x,y,\theta,0)>0$ for all θ, implying that $\theta^*(0^+)\varepsilon\ \mathrm{argmax}_\theta \mathrm{W_{BB}}(\mathrm{x,y}$ and $W_{BB\theta}(x,y,\theta^*(0^+),0)=0$. Since

$$\alpha\rho W_{BB}(x,y)=4(\theta-\theta^2/2)\left(F(x)^2_F(y)^2+\gamma(G(x)^2+G(y)^2)\right) \tag{31}$$

$\theta^*(0^+)=1$. Therefore, when the prize is small, it is optimal, in the sense of maximizing total social surplus, to give nearly all of the social benefits to the innovator.

Note that this does not contradict our earlier result that innovation is excessive whenever the prize equals the benefit, just that the difference between the opitmal prize and the social benefit goes to zero faster than the social benefit. This is not surprising since it just says that the externalities due to the competition over the rents fall more rapidly than B as B goes to zero. The primary point of this exercise is to illustrate how to determine the limit.

Second, further expansion of the social value function and application of l'Hospital's rule shows that the optimal θ falls more rapidly as B increases when $F(x)$ and $G(x)$, the probability of an immediate success from the current position (assuming the players begin at the same position), rises. This implies that the shorter the race, the smaller should be the winner's share under competition. Since the details entail only repeated applications of the foregoing calculations, they are omitted here.

Another crucial aspect of patent policy is the stage at which a patent is granted. A patent may be granted before final and complete success is achieved. In fact, in the existing patent system, a patent is granted when a description of an invention has been completed, before the development stages leading to a workable and commerical prototype have been achieved. This may be socially optimal if the effort of followers is so excessive and wasteful that it is better to force them out of the race, bearing the possible inefficiencies that may result when an innovator is given the monopoly early. In our model, this can be modeled by assuming that a patent is granted to the first firm which crosses $c \leq 0$. If $c = 0$, the firm must complete the project before acquiring a patent worth P. If $c < 0$, then a firm receives a patent at c and may finish development without any competition.

Proceeding as in the $c = 0$ case, we find that the equilibrium value function for the players solves

$$\begin{aligned} 0= \;& \left(\textstyle\int_x^c V(s,y)f(s,x)ds + \int_c^0 M(s)f(s,x)ds + PF(x) - V(x,y)\right)^2/2\alpha \\ &+ \left(\textstyle\int_y^c V(s,x)f(s,y)ds + \int_c^0 M(s)f(s,y)ds + PF(y) - V(y,x)\right) \\ &\quad \times \left(\textstyle\int_y^c V(x,s)f(s,y)ds - V(x,y)\right)/\alpha - \rho V(x,y) \end{aligned} \tag{32}$$

where $M(\cdot)$ is the monopoly value function computed in section 2 with the extension to two instruments, u and v or w and z. If we expand (32) as before for the case of a small social benefit and prize, we find that when P is small the loss in $V(x,y)+V(y,x)$, the social value function if P and B are equal, when $c < 0$ compared to $c = 0$ is approximated by

$$F(y)\int_c^0 g^1(x,s)f(s,y)ds + F(x)\int_c^0 g^1(y,s)f(s,x)ds > 0 \tag{33}$$

Hence, the major factor is that if $c < 0$, the contest is ended early and the resulting loss in total effort is excessive relative to the cost savings.

Theorem 7 summarizes our findings concerning optimal policy.

Theorem 9. *When B is small, the optimal policy is to award a prize only when the race is completely won and the prize should be nearly the entire social value of the innovation. Furthermore, the closer the innovators are to final success when competition begins, the less should be their share in the social benefit.*

While these conclusions are surely not globally true, we have shown their validity for an open set of problems. More importantly, we have shown how to address these questions for that collection. Other exercises, such as the impact of suboptimal innovation resource allocation on the optimal prize, can be conducted by straightforward examination of the higher-order terms of our expansion for W, the social planner's objective. In the interest of space, we leave such extensions to the reader.

9. Generalizations

There are many other exercises which could have been pursued, but were not because the one examined above most clearly illustrates the general approach advanced here. To indicate that this approach is not too specialized, we will now discuss some other possible exercises.

All models with closed-form solutions are degenerate in some sense. When we use them we hope that the effects from which we abstract are not important. Take, for example, the model used by Loury and Reinganum. While it yields closed-form solutions for the quadratic cost specification, it abstracts from the possibility of intermediate stages, our focus here. Recall that our model with F and G equal to constant functions is exactly that model. To examine the importance of intermediate stages on the nature of equilibrium, we could have assumed that F and G deviated slightly from constant functions. This alternative would have allowed us to determine the nature of equilibrium for arbitrary prize but with only a small deviation from the implicit stage-independence of Reinganum's analysis.

Another possible generalization is allowing intermediate payments. While in some of these earlier models there were prizes for intermediate success, we assumed no such intermediate prizes nor social benefits. However, the analysis conducted above could also allow intermediate payoffs since nothing we did used the absence of intermediate payoffs in an essential fashion; we focussed on the more simple payoff structure since our purpose was to present a robust analysis of the positional dynamics among competitors for one kind of race. A more general analysis with intermediate payoffs could generate insights, for example, into strategic implications of the learning curve; one approach would be to approximate the slow-learning case by knowing the solution to the no-learning case. However, we leave such an analysis to another study.

While this is certainly not an exhaustive list, it does argue for the assertion that the approach of this paper is useful in examining the robustness of simple models generally, allowing us to add some otherwise intractable element to the analysis of a problem. While our analysis got started by examining the trival case of no payoff, generally one can begin with any tractable case. We hope that this exercise has not only generated interesting results about the nature of innovation rivalry, but also demonstrated that perturbation analysis generally can be a valuable tool for dynamic strategic analysis.

8. **Conclusion**

We have analyzed a simple closed-loop subgame perfect model of multi-stage innovation. We found the usual result of excessive innovative effort when the prize equals the social value. Under the assumption that the net social value of innovation is small, we have also found that there will be excessive risk-taking, that at any moment the following firm is a less efficient innovator relative to the leader, that the prize to the innovator should nearly equal social benefits, and that the competition should not be ended before one of the competitors has succeeded completely. While these results have obvious limitations on their generality, the do tell us that the contrary propositions cannot be generally true. While many of the results, e.g., the excessive investment when the prize equals the social benefit, follow naturally from the fact that these subgame perfect equilibra are close to some open-loop equilibria others, in particular the computation of the equilibrium reactions, are specifc to the subgame-perfect solution. They have therefore given us a peek into the nature of subgame perfect equilibrium in such innovation models.

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