Computational Methods for Dynamic Equilibria with Heterogeneous Agents

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Computational Methods

- A Natural Part of Econometrics as defined by Frisch
  - ".. as long as we confine ourselves to statements in general terms about one economic factor having an effect on some other factor, almost any sort of relationship may be selected.."
  - Economic analysis requires a comparison of magnitudes of complex interactions operating in all directions.
  - "Mathematics is indispensable ... necessary for discussing issues safely and consistently"
- Computational methods are necessary to examine the complex relationships in modern dynamic stochastic models.
- Early methods
  - Based on economically intuitive stories, such as tattonnement
  - Suffered from many problems - poor convergence properties, low accuracy
- The past decade
  - A substantial influx of mathematical technique
  - Development of faster and more reliable methods for dynamic economic models
- Our objective
  - Give an overview of some of the key ideas
  - Give examples of their advantages.

Perfect Foresight Models

- Canonical model

$$g(t, x_t, x_{t+1}) = 0,\ t = 0, 1, 2, ... \tag{1}$$

- Fair-Taylor (Ecm., 1983)
  - A Gauss-Jacobi scheme: Given a guess for x_{t+1}, use time t equation to find new guess for x_t
  - Slow, possibly nonconvergent; loose accuracy

- L-B-J (see Boucekkine, (JEDC, 1995), and Juillard et al (JEDC, 1998))
  - Jacobian is sparse since time t equation depends on only (x_t, x_{t+1})

$$J(x) = \begin{pmatrix} g_2(1,x_1,x_2) & g_3(1,x_1,x_2) & 0 & 0 & \cdots \\ 0 & g_2(2,x_2,x_3) & g_3(2,x_2,x_3) & 0 & \cdots \\ 0 & 0 & g_2(3,x_3,x_4) & g_3(3,x_3,x_4) & \cdots \\ 0 & 0 & 0 & g_2(4,x_4,x_5) & \cdots \\ 0 & 0 & 0 & 0 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix} \tag{2}$$

  - Use sparse Newton method from literature on solving large systems
  - Far faster than Fair-Taylor

- Gilli - Paulletto (JEDC, 1998)
  - Inverting even sparse matrices is difficult
  - Use Krylov methods to invert Jacobian in a Newton scheme
- Judd (2000) - parametric path method
  - Observe that solutions tend to be smooth in t
  - Approximate x_t with a polynomial-exponential expression in t

$$\left(\sum_{i=0}^{N} a_i t^i\right) e^{-\lambda t} + x^* \left(1 - e^{-\lambda t}\right) \tag{3}$$

  - Use orthogonal polynomial, integration ideas to find a good subset of $M > N$ equations

$$g(t_j, x_{t_j}, x_{t_j+1}) = 0,\ j = 1, 2, ...M \tag{4}$$

  - Use this subset to solve for coefficients (use overidentification method)
- Conclusion: Application of methods for solving large nonlinear systems have produced faster and more reliable solution methods for perfect foresight models.

Projection Methods

- The mathematical literature on projection methods (also called method of weighted residuals) provides us with a framework for describing many algorithms for solving dynamic economic models.

- Simple One-Agent Example

$$\begin{aligned} u_i'(C(k,\theta)) &= \beta\, E\left\{u'(C(k^+,\tilde{\theta}))R(k^+,\tilde{\theta})\ \mid \theta\right\} \\ k^+ &= F(k,\theta) - C(k,\theta) \end{aligned}$$

- Step 0: Choose function to approximate
  - $C(k,\theta)$ is natural here
  - Wright and Williams (1982): approximate conditional expectation function, a smoother function.

- Approximate $C(k,\theta)$
  - Polynomial in (k,θ) or $(\log k, \log\theta)$ or etc.
  - Orthogonal polynomials

$$\widehat{C}(k,\theta) = \sum_{i=0}^{n} a_i \varphi_i(k,\theta) \tag{5}$$

- Define residual of the approximation

$$R(k,\theta;a) = u_i'(\widehat{C}(k,\theta;a)) - \beta\, E\left\{u'(\widehat{C}(k^+,\tilde{\theta};a))R(k^+,\tilde{\theta})\ \mid \theta\right\} \tag{6}$$

- Replace conditional expectation with a numerical approximation
  - Gaussian quadrature, monomial rules
  - Monte Carlo, quasi-Monte Carlo

- Define (and numerically approximate) projection conditions relative to a set of test functions $p_i(\cdot)$:

$$P_i(a) \equiv \langle R(\cdot\,;\,a)\,, p_i(\cdot)\rangle_2, i = 1, \cdots, n. \tag{7}$$

- Solve projection conditions

$$P_i(a) \equiv 0, i = 1, \cdots, n. \tag{8}$$

  - Tatonnement or learning story
  - Solve backwards from some fixed date
  - Newton's method
  - Homotopy or Scarf method

- Test candidate solution a^* by computing some test norm with high accuracy

  - Judd (1992) computes $\|R(k, \theta; a)\|_p$ using deterministic quadrature for $p = 1, 2, \infty$
  - den Haan and Marcet (1994) computes moments of $R(k_t, \theta_t; a^*)$ in Monte Carlo simulations

Models with Many Agents

- Simple Two-Agent Growth Example, common preferences

$$
\begin{aligned}
u_i'(C^i(k_1,k_2)) &= \beta\, u'(C^i(k_1^+,k_2^+))R(k^+), \quad i=1,2 \\
k_i^+ &= Y^i(k) - C^i(k), \quad i=1,2 \\
Y^i(k) &= k_i R(k) + w(k), \quad i=1,2 \\
R(k) &= F'(k_1+k_2) \\
w(k) &= F(k_1+k_2) - (k_1+k_2)F_k(k_1+k_2)
\end{aligned} \tag{9}
$$

- Same procedure as with one agent, just with more Euler equations and more consumption functions
  - Tensor product approximation of $C^i(k_1,k_2,\theta)$

$$
C^i(k;a) \doteq \sum_{j_1=0}^{J_1}\sum_{j_2=0}^{J_2} a^i_{j_1 j_2 m}\, k_1^{j_1}\, k_2^{j_2}\,, \quad i=1,2 \tag{10}
$$

  - complete polynomial approximation

$$
C^i(k;a) \doteq \sum_{\substack{0\le j_1+j_2\le d \\ 0\le j_1,j_2\le d}} a^i_{j_1 j_2}\, k_1^{j_1}\, k_2^{j_2}, \quad i=1,2 \tag{11}
$$

- Suppose we had m agents (See Gaspar and Judd (1997))
  - c_i depends on k_i and other k's but symmetrically, so

$$C^m(k;a) \doteq \sum_{\substack{0\le i+j\le d \\ 0\le i,j\le d}} a_{i,j}\; k_m^i \; \varphi_j(k_1,..,k_{m-1},k_{m+1},...,k_n) \tag{12}$$

    where $\phi_j(k_{-i})$ is a symmetric polynomial

  - Linear and quadratic symmetric polynomials are

$$\begin{array}{l} x+y+...+z \\ x^2+y^2+...+z^2 \\ (x+y+...+z)^2 \end{array}$$

    and cubic ones are

$$\begin{array}{l} x^3+y^3+...+z^3 \\ x^2y+x^2z+...+y^2z+... \\ (x+y+...+z)^3 \end{array}$$

- Suppose we had a continuum of agents.
  - Krusell and Smith (1997) use approximation

$$C_i(k;a) \doteq \sum_{\substack{0\leq i+j\leq d \\ 0\leq i,j\leq d}} a_{i,j}\; k_m^i\; f(\mu,\sigma^2) \tag{13}$$

    which is a symmetric polynomial with a continuum of variables
  - den Haan (1997) parameterizes the distribution of wealth more flexibly than just mean and variance

$$\begin{aligned} F(k) &\doteq \exp\left(\sum_{i=0}^{N} b_i k^i\right) \\ C_i(k;a) &\doteq \sum_{\substack{0\leq i+j\leq d \\ 0\leq i,j\leq d}} a_{i,j}\; k_m^i\; f(b) \end{aligned}$$

    den Haan (2000) shows that more flexibility is needed sometimes.

- Define and solve projection conditions

$$P_i(a) \equiv 0, i = 1, \cdots, n. \tag{14}$$

- Test candidate solution a^* by computing some test norm with high accuracy

Incomplete Asset Markets: A Competitive Example

- Basic Problem
  - Two Agents, with idiosyncratic Markovian endowment risk
  - Two assets (a stock and a bond)
- Heaton and Lucas (1996)
  - Discretized asset space (30×30)
  - Solve backwards in time
  - Converged, computed two-digit accuracy (on average) in market-clearing price
- Marcet and Singleton (1999)
  - Parameterized conditional expectation with low-order polynomial
  - Use learning iteration
  - Report convergence problems
- Judd, Kubler, and Schmedders (1999)
  - Use cubic splines to approximate consumption, investment, and price laws
  - Use homotopy methods to compute state-specific equilibria
  - Backward-in-time method, converged - relatively rapidly
  - Euler equation residuals on order of 1 in 10,000

- General problems
  - Must impose some kind of asset constraint
    - Portfolios tend to wander in equilibrium with incomplete assets
    - Shorting constraints are natural but they create kinks in policies and conditional expectations
  - Arbitrage problems
    - Algorithms search price space
    - Some guesses imply arbitrage opportunites, undefined demand
  - Ill-conditioning problems
    - Bonds and equities tend to be good substitutes
    - Jacobian of demand function is ill-conditioned
    - Need to use algorithm which can handle ill-conditioning - homotopy methods, TENSOLVE

Time Consistency Problems: A Strategic Example

- Basic Problem
  - Government's future actions affect current private behavior
  - Government cannot commit
- General approach
  - Strategic agents have value functions, $V^i(x)$, and Bellman equations

$$V^i(x,u) = \max_{u_i} \pi(u,x) + \beta V^i(f(x,u)),\ i = 1,2 \tag{15}$$

  - Competitive agents have policy functions for decision rules given by Euler equations which determine the law of motion $f(x,u)$
  - Method: parameterize value functions and competitive agents decision rules
- Early Computational Examples
  - Gov't oil storage vs. private storage: Williams and Wright (Bell, 1982)
  - Approximate private and gov't decision rules: WW used degree 6 polynomials.
  - Similarly in Kotlikoff-Shoven-Spivak (1986) computation of strategic bequests.
  - Low accuracy level (two digits) and slow
- Recent examples of time consistency computations
  - Miranda and Rui (1996): strategic storage game between producers of a commodity
  - Ha and Sibert (1997): strategic capital taxation game between countries with capital flows
  - Both used orthogonal polynomial and collocation methods to achieve higher accuracy in more complex models

Perturbation and Asymptotic Methods

- General Problem

$$\begin{aligned} E\{g(x_t, y_t, x_{t+1}, y_{t+1}, \epsilon)|x_t\} &= 0 \\ x_{t+1} &= F(x_t, y_t, \epsilon z_t) \end{aligned} \quad (16)$$

with solution $Y(x, \varepsilon)$

$$E\{g(x, Y(x,\epsilon), F(x, Y(x,\epsilon), \epsilon z), Y(F(x, Y(x,\epsilon), \epsilon z), \epsilon)|x\} \doteq 0 \quad (17)$$

- Compute steady state

$$\begin{aligned} g(x^*, y^*, x^*, y^*, 0) &= 0 \\ x^* &= F(x^*, y^*, 0) \end{aligned}$$

- Construct Taylor series approximation

$$Y(x,\epsilon) \doteq y^* + Y_x(x,0)(x - x^*) + Y_\epsilon(x,0)\epsilon + (x - x^*)' Y_{xx}(x,0)(x - x^*) + ... (18)$$

- Magill and usual linearization method
  - Replaced nonlinear problem with a LQ example with same local deterministic dynamics
  - RBC models use special case of Magill exposited in Kydland-Prescott
  - Certainty equivalent, hence only valid for variance approximations, not means or utility. Computes only

    $$Y(x,\epsilon) \doteq y^* + Y_x(x,0)(x - x^*) \quad (19)$$

    ignoring first order term $Y_\epsilon(x,0)\epsilon$ as well as higher-order terms.
  - Only local validity

- Want higher-order approximations
  - Linear approximations often do poorly away from deterministic steady state
  - Linear approximations implicitly assume quadratic utility, which has unappealing properties: e.g., increasing ARA
- Mathematics literature
  - Can compute high-order Taylor series approximation
  - Can compute certainty nonequivalent methods
- Recent applications
  - Judd and Guu (1993) show how to use perturbation methods to solve simple one sector optima growth problems.
  - Gaspar and Judd (1997) apply perturbation methods to optimal control problems.
  - Zadrozny and Chen (2000), and Collard, Feve, and Juillard (2000) examine general rational expectations models.
  - Mrkaic (1998) uses perturbation methods to evaluate some econometric procedure.

Bifurcation Methods for Small Noise Portfolio Problems

- Suppose there are assets
  - Demand at $\sigma^2 = 0$ is not well-defined since all assets are substitutes
  - "Deterministic steady state" is not well-defined
  - Linear-quadratic examples do not do well - Kim-Kim (1999)
  - Campbell - proposes an ad hoc procedure

- Bifurcation methods
  - Parameterize simple one-period portfolio problem

$$Z = R + \epsilon z + \epsilon^2 \pi, \tag{20}$$

  - First-order condition

$$0 = E\{u'\left(R + \omega(\epsilon z + \epsilon^2 \pi)\right)\ (z + 2\epsilon\pi)\} \equiv\ G(\omega, \epsilon). \tag{21}$$

  - ω is indeterminate at $\epsilon = 0$ since

$$0 = G(\omega, 0), \forall\, \omega. \tag{22}$$

  - Implicit differentiation implies that

$$0 = G_\omega\, \omega' + G_\omega. \tag{23}$$

    but appears invalid since $G_\omega = 0$
  - L'Hospital's rule says that if $G_\omega(\omega(0), 0) = 0$ then

$$\omega'(0) = -\frac{G_{\epsilon\epsilon}(\omega(0), 0)}{G_{\omega\epsilon}(\omega(0), 0)} \tag{24}$$

  - Bifurcation theory says choose $\omega(0)$ so that $G_\omega(\omega(0), 0) = 0$, and use L'Hospital's rule (and multidimensional extensions) to compute Taylor series of $\omega(\varepsilon)$
  - Can be used to compute asset market equilibrium with small noise and multiple investors - see Judd and Guu (2000)

Conclusions

- There is steady progress in computing equilibria in markets with several agents
- Key tools
  - Exploitation of approximation theory
  - Methods for solving large systems
- Tractable problems
  - Dynamic markets with complete or incomplete asset markets
  - Strategic interactions, such as time inconsistency problems
- Further progress is likely
  - Integration of symbolic and numerical methods make perturbation methods more tractable
  - More advanced integration methods make higher-dimensional problems more tractable