A solution method for incomplete asset markets with heterogeneous agents*

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Abstract

This paper examines a dynamic, stochastic endowment economy with two agents and two financial securities. Markets are incomplete and agents can have heterogeneous tastes. We develop a new computational method to solve the dynamic general equilibrium model. We allow for various forms of portfolio constraints, transaction costs, and portfolio penalties.

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1 Introduction

General equilibrium analysis provides an ideal framework to study prices in financial markets. Early examples of equilibrium asset pricing models include the Lucas asset pricing model (Lucas (1978)), a model of an infinite horizon economy with a single agent and Markovian exogenous shocks. While the model is theoretically interesting because there exists a stationary equilibrium which can be easily computed (see Judd (1992)) it fares poorly in explaining observed security prices (see for example Mehra and Prescott (1985), Schwert (1989)). With the development of the general equilibrium theory with incomplete asset markets in the 1980's it is now well understood how to extend the Lucas-model to incorporate heterogeneous agents (see Duffie et al. (1994), Magill and Quinzii (1996)) and one can investigate to what extent the joint hypothesis of agents' heterogeneity and incomplete markets helps to explain observed prices (see for example Mankiw (1986), Telmer (1993), Constantinides and Duffie (1996), Heaton and Lucas (1996), Krusell and Smith (1997) or den Haan (1997)).

The primary difficulty in examining these models is that generally there do not exist stationary equilibria where the endogenous variables are functions of the exogenous shock alone. The wealth distribution affects equilibrium prices and portfolio holdings, and in some cases one would also expect lagged prices and consumption to be part of the minimal sufficient state space (see Duffie et al. (1994)). There are no closed-form solutions for equilibrium prices, portfolios and consumption plans and instead one has to rely on computational methods to analyze equilibria. In computing equilibrium prices and asset holdings in infinite horizon models with incomplete markets we face a variety of difficulties – in particular one faces a curse of dimensionality as the dimension of the endogenous state space increases as the number of agents or assets increases.

There are several papers that develop computational strategies for infinite horizon incomplete markets models. Den Haan (1997) assumes that there is a single asset and there are infinitely many ex ante identical agents and parameterizes the cross sectional distribution of asset holdings, which he assumes to provide a sufficient endogenous state, with a flexible functional form. Krusell and Smith (1997) follow a similar strategy only that they have two (short-lived) assets and use the cross-sectional distribution of wealth as a state variable. This approach has the disadvantage that one cannot incorporate heterogeneous tastes (only random variation in tastes as in Krusell and Smith) and that it seems almost hopeless to extend it to a model with several (long-lived) assets.

Lucas (1994) and Heaton and Lucas (1996) consider a model with two agents and two securities and assume that exogenous shocks together with agents' current period portfolio holdings form a sufficient statistic for the future evolution of the economy. They discretize the state space (the allocation of assets) and develop a Gauss-Jacobi scheme which, in each state, finds a pattern of trades which nearly clears the market. Because of the discrete state space (they allow for only 30 different values for an agents' holdings in each security) this approach possibly yields large approximation errors. They report average (not maximum) errors in market clearance of up to 0.84 percent. Since Heaton and Lucas have a discrete endogenous state space they cannot easily improve their approximation. In particular, they allow trade to occur only once a year – more precisely, they assume a period discount factor of 0.95. Zhang (1997 a,b) examines a model with a single asset and he also discretizes the state space, but he uses a linear interpolation rule to approximate the equilibrium functions between grid points.

In this paper we develop an algorithm for a model with two agents and two securities using standard methods from numerical nonlinear functional analysis. Following the applied literature, we assume that portfolio holdings alone constitute a sufficient state space. The innovation of our algorithm is to approximate the equilibrium functions with two-dimensional splines, to form the Euler equations for bond and stock holding, and to use collocation to solve the resulting nonlinear systems of equations for the B-spline coefficients. Our maximum Euler equation errors lie in the range of one dollar per $ 100,000 to $ 1,000,000 of consumption. These errors indicate that for the class of economies we consider lagged prices and consumption do not need to be included in the state space. We show that smooth approximations of the equilibrium functions are needed in order to discover important qualitative features of equilibrium. We show that for the class of models we consider there is a one-to-one equilibrium relationship between wealth and the portfolio holdings of the first agent. We are also able to compute equilibria for models with heterogeneous tastes. Furthermore, because of our smooth approximation of trading strategies our method can potentially be used to allow more frequent trading, an important improvement (with a discrete state space, the model cannot be calibrated to high-frequency data because per period trading becomes to small to be captured in a coarse grid).

The paper is organized as follows. In Section 2 we present the model under consideration. Section 3 develops the algorithm and gives a detailed discussion of both economically and computationally motivated aspects of the implementation. Section 4 presents numerical examples.

2 The Economic Model

We examine a Lucas asset pricing model with heterogeneous agents. This model is a special case of the infinite horizon incomplete markets economy discussed in Hernandez and Santos (1996) in that it assumes that endowments and dividends are Markovian. Duffie et al. (1994) provide a general theoretical treatment of the model we use.

There is a time-homogeneous Markov process of exogenous income states $(y_t)_{t\in\mathbb{N}}$ which are assumed to lie in a discrete set $Y = \{1, 2, \ldots, S\}$. The transition matrix is denoted by $\mathbf{P}$. There are two agents and there is a single perishable good at each state. Agent h's individual endowment in period t given income state y is assumed to be a function $e^h : Y \to \mathbb{R}_{++}$ depending on the exogenous income state alone. In order to transfer wealth across time and states agents trade in securities. There is a (short-lived) riskless bond in each period paying one unit of the good in each state of the next period and there is a long-lived asset in unit net supply paying a dividend $d : Y \to \mathbb{R}_+$ each period. We denote agent h's portfolio in period t by $\theta_t^h = (\theta_t^{hb}, \theta_t^{hs}) \in \mathbb{R}^2$ and his initial endowment of the stock prior to time 0 by θ_{-1}^{hs}. Let $e_t = e^1(y_t) + e^2(y_t) + d(y_t)$ denote the aggregate endowment in period t.

Each agent h has von-Neumann-Morgenstern preferences which are defined by a strictly monotone C^2 concave utility function $u_h : \mathbb{R}_{++} \to \mathbb{R}$ that possesses the Inada property, that is, $\lim_{x\to 0} u'(x) = \infty$, and a discount factor $\beta_h \in (0, 1)$. For any consumption sequence $c = (c_0, c_1, c_2, \ldots)$ the associated utility for agent h is therefore:

$$U_h(c) = E\left\{\sum_{t=0}^{\infty} \beta_h^t u_h(c_t)\right\}.$$

2.1 Transaction Costs

It seems realistic to assume that there is a real cost in acquiring financial assets. The following specification of transaction costs is from Heaton and Lucas (1996, Section IV D).

At each date t an agent h pays transaction costs of $\omega(\theta_{t-1}^h, \theta_t^h)$. We assume that ω has the functional form

$$\omega(\theta_{t-1}, \theta_t) = \tau^b (q_t^b \theta_t^b)^2 + \tau^s (q_t^s (\theta_t^s - \theta_{t-1}^s))^2,$$

where τ^b, τ^s are constants.

The assumption of strictly convex costs is unrealistic but it is needed to ensure that agents face a differentiable and convex programming problem.

2.2 Portfolio Restrictions

In order to define equilibrium one has to rule out the possibility of an infinite accumulation of debt and introduce constraints on agents' net wealth holdings. Furthermore, with long-lived stocks and incomplete markets one faces the usual existence problem which arises in GEI models with real assets and is caused by a discontinuity in the demand function. Magill and Quinzii (1996) and Hernandez and Santos (1996) describe possible debt constraints and give proofs of generic existence. For our purposes, however, it is much more useful to directly restrict the portfolio positions agents are allowed to hold. In this paper we examine three different restrictions on portfolio holdings. In Section 2.6 we will examine how short-sale constraints relate to debt constraints and in Section 4 we show how different short-sale constraints affect the equilibrium outcomes.

2.2.1 Short-Sale Constraints

The easiest way to obtain a bounded set of portfolios is to impose a short-sale constraint, see Heaton and Lucas (1996, 1997). Short sales can be constrained through a priori specified fixed exogenous lower bounds on the portfolio variables, that is,

$$\theta^{hb} \geq -B^{hb} \text{ and } \theta^{hs} \geq -B^{hs},$$

where $B^{hb}, B^{hs} \geq 0$. Note that we define the bounds on short sales as agent dependent since it is certainly realistic to assume that an agent's income influences how much he can borrow.

2.2.2 Penalties on Portfolios

A second – and perhaps more realistic – approach is to assume that agents are allowed to hold portfolios of any size but get penalized for large portfolio holdings; the intuition behind such a model assumption is that there are costs associated with large short positions and in a simplification we model them as penalties to agents' utilities; when these penalties get sufficiently large, agents will avoid extreme positions. The advantage of utility penalties on large short positions is that this restriction does not constitute an a priori exogenous constraint on short sales. Instead, the penalties lead to endogenous avoidance of short sales depending on how much agents desire large short positions.

We use a penalty function of the form

$$\rho^h(\theta) = \kappa^b \min(0, \theta^{hb} - L^{hb})^4 + \kappa^s \min(0, \theta^{hs} - L^{hs})^4$$

where $\kappa^a \geq 0, a \in \{b, s\}$ and $L^{ha} \leq 0$. Note, there is no punishment for large long positions. If κ^a is sufficiently large the penalty function almost acts like a hard short-sale constraint on the corresponding asset $a \in \{b, s\}$. For a more general description of the model it suffices that ρ is a convex function satisfying $\rho(\theta) \to \infty$ as $|\theta| \to \infty$. The portfolio penalties lead our agents to have utility functions over consumption and portfolio holdings of the form

$$V_h(c, \theta) = U_h(c) - E\left\{\sum_{t=0}^{\infty} \beta^t \rho^h(\theta_t)\right\}.$$

2.3 Competitive Equilibrium

We define an economy $\mathcal{E} = (\mathbf{e}, \mathbf{d}, \mathbf{P}, \beta_1, \beta_2, u_1, u_2, \tau^b, \tau^s) \in \mathrm{I\!R}_{++}^{2S+2S+SS+2} \times \mathcal{U} \times \mathcal{U} \times \mathrm{I\!R}_{+}^2$, where

$$\mathbf{e} = \begin{pmatrix} e^1(1) & e^2(1) \\ \vdots & \vdots \\ e^1(S) & e^2(S) \end{pmatrix}, \quad \mathbf{d} = \begin{pmatrix} 1 & d^s(1) \\ \vdots & \vdots \\ 1 & d^s(S) \end{pmatrix}$$

are the matrices of individual endowments and security dividends, respectively, and $\mathcal{U}$ is the set of utility functions satisfying the assumptions on monotonicity and concavity. If we consider only economies with portfolio penalties we write $\mathcal{E}^P$; for economies with short-sale constraints we denote the feasible portfolio set of agent h by K^h and a generic economy of this type by $\mathcal{E}^K$.

The notion of a competitive equilibrium for economies $\mathcal{E}^P$ with portfolio penalties now defined as follows:

Definition 1 *A competitive equilibrium for an economy $\mathcal{E}^P$ is a collection of portfolio holdings $\{(\theta_t^1, \theta_t^2)\}$ and asset prices $\{q_t\}$ satisfying the following conditions:*

(1) $\theta_t^{1b} + \theta_t^{2b} = 0$ *and* $\theta_t^{1s} + \theta_t^{2s} = 1$ *for all* t.

(2) For each agent $h: (c, \theta^h) \in \arg\max V_h(c, \theta)$ *s.t.*
$$c_t = e_t^h + \theta_{t-1}^b + \theta_{t-1}^s(q_t^s + d_t) - \theta_t q_t - \omega(\theta_{t-1}, \theta_t)$$

For the definition of equilibrium for an economy $\mathcal{E}^K$ with short-sale constraints condition (2) must be restated as

(2') For each agent $h: (c, \theta^h) \in \arg\max U_h(c)$ s.t.
$$c_t = e_t^h + \theta_{t-1}^b + \theta_{t-1}^s(q_t^s + d_t) - \theta_t q_t - \omega(\theta_{t-1}, \theta_t)$$
$$\theta_t \in K^h$$

Note that by Walras' law condition (1) ensures that good markets clear, that is, aggregate consumption equals aggregate endowments minus the resources 'burned' for transactions.

For the marginal transaction costs and holding penalties we define the notation

$$
\begin{aligned}
\omega_{s1} &= \frac{\partial\omega(\theta_{t-1},\theta_t)}{\partial\theta^s_{t-1}} &= -2\tau^s(q^s_t)^2(\theta^s_t-\theta^s_{t-1}) \\
\omega_{s2} &= \frac{\partial\omega(\theta_{t-1},\theta_t)}{\partial\theta^s_t} &= 2\tau^s(q^s_t)^2(\theta^s_t-\theta^s_{t-1}) \\
\omega_b &= \frac{\partial\omega(\theta_{t-1},\theta_t)}{\partial\theta^b_t} &= 2\tau^b(q^b_t)^2(\theta^b_t) \\
\rho^h_b &= \frac{\partial\rho^h(\theta_t)}{\partial\theta^b_t} \\
\rho^h_s &= \frac{\partial\rho^h(\theta_t)}{\partial\theta^s_t}
\end{aligned}
$$

Agent h's first-order conditions for optimization are:

$$
\begin{aligned}
(q^b_t+\omega_b)u'_h(c_t)-\rho^h_b &= \beta_h E_t(u'_h(c_{t+1}))) && (1) \\
(q^s_t+\omega_{s2})u'_h(c_t)-\rho^h_s &= \beta_h E_t\left\{(q^s_{t+1}+d_{t+1}-\omega_{s1})u'_h(c_{t+1})\right\} && (2)
\end{aligned}
$$

Under our assumptions on preferences the Euler equations of both agents together with the market-clearing conditions are necessary and sufficient for equilibrium.

2.4 Stationary Equilibria

In order to compute an equilibrium for an infinite horizon model it is necessary to focus on equilibria which are dynamically simple; it must be possible to describe the state of the system by a small number of parameters which provide a sufficient statistic for the evolution of the system. Contrary to the complete-markets model the state space for our models will also include endogenous variables since they obviously influence the evolution of security prices in equilibrium. In particular one expects the distribution of agents' portfolio holdings to influence equilibrium prices. We denote the set of agent 1's possible equilibrium portfolio holdings by $\Theta \subset \mathbb{R}^2$. Note that in the case of short-sale constraints Θ is compact.

Duffie et al. (1994) define an endogenous state space Z_{THME} which includes current and last period portfolio holdings, the asset prices of the current period, and the agents' consumptions in the current period. They need this large state space in order to be able to convexify the equilibrium correspondence with sunspots. However, for computational purposes the approach cannot be used.

2.5 Recursive Equilibria

The usual assumption in the applied literature (see for example Telmer (1993) or Heaton and Lucas (1996)) is that the exogenous state and the agents' portfolio holdings alone constitute a sufficient state space for the evolution of the infinite horizon economy. Moreover, the existence of continuous policy functions f and price functions g is postulated, which map last period's portfolio holdings and the current exogenous income state into the current period portfolio holdings and asset price, respectively. For our model these assumption mean that Θ is the endogenous state space and that there exist a continuous function $f : Y \times \Theta \to \Theta$ which determines agent 1's optimal portfolio choice in the current period given an exogenous income state $y \in Y$ and portfolio holding $\theta_- \in \Theta$. Similarly, a continuous price function $g : Y \times \Theta \to \mathbb{R}^2_{++}$ maps the current period exogenous income state $y \in Y$ and agent 1's portfolio holding $\theta_- \in \Theta$ into the current period prices of the securities.

We formalize the notion of a recursive equilibrium for an economy $\mathcal{E}^P$. Let C be the set of continuous functions and let and U be the set of all continuous functions $f : Y \times \Theta \to \Theta$ where

$$f = ((f_1^b, f_1^s), (f_2^b, f_2^s), \ldots, (f_S^b, f_S^s)),$$

and let V be the set of all continuous functions $g : Y \times \Theta \to \mathbb{R}^2_{++}$ where

$$g = ((g_1^b, g_1^s), (g_2^b, g_2^s), \ldots, (g_S^b, g_S^s)).$$

Define

$$c^1(y, \theta_-) = e^1(y) + \theta_-^b + \theta_-^s(g(y, \theta_-) + d(y)) - f(y, \theta_-)g(y, \theta_-) - \omega(\theta_-, f(y, \theta_-)).$$

Then we have,

$$c^2(y, \theta_-) = e(y) - c^1(y, \theta_-) - 2\omega(\theta_-, f(y, \theta_-)).$$

Let $W \subset U \times V$ be the set of all f and g such that $c^1(y, \theta_-) > 0$ and $c^2(y, \theta_-) > 0$ for all $y \in Y$ and all $\theta_- \in \Theta$. For all $y \in Y$ and $\theta_- \in \Theta$ we define a functional $F : W \to C \times C$ as follows:

$$\begin{aligned}
F_{1b}^y(f, g)(\theta_-) &= (g^b(y, \theta_-) + \omega_b^f)u_1'(c^1(y, \theta_-)) - \rho_b^h - \beta_1 E_y[u_1'(c^1(\tilde{y}, f(y, \theta_-)))] && (3)\\
F_{1s}^y(f, g)(\theta_-) &= (g^s(y, \theta_-) + \omega_{s2}^f)u_1'(c^1(y, \theta_-)) - \rho_s^h - \\
&\quad \beta_1 E_y[(g^s(\tilde{y}, f(y, \theta_-)) + d(\tilde{y}) - \tilde{\omega}_{s1}^f)u_1'(c^1(\tilde{y}, f(y, \theta_-)))] && (4)\\
F_{2b}^y(f, g)(\theta_-) &= (g^b(y, \theta_-) + \omega_b^f)u_2'(c^2(y, \theta_-)) - \rho_b^h - \beta_2 E_y[u_2'(c^2(\tilde{y}, f(y, \theta_-)))] && (5)\\
F_{2s}^y(f, g)(\theta_-) &= (g^s(y, \theta_-) + \omega_{s2}^f)u_2'(c^2(y, \theta_-)) - \rho_s^h - \\
&\quad \beta_2 E_y[(g^s(\tilde{y}, f(y, \theta_-)) + d(\tilde{y}) - \tilde{\omega}_{s1}^f)u_2'(c^2(\tilde{y}, f(y, \theta_-)))] && (6)
\end{aligned}$$

where

$$\omega_b^f = \omega_b|_{\theta^b = f^b(y, \theta_-)},\ \tilde{\omega}_{s1}^f = \omega_{s1}|_{\theta_-^s; \theta^s = f^s(y, \theta_-)},\ \omega_{s2}^f = \omega_{s2}|_{\theta^s = f^s(y, \theta_-)},$$

and

$$\rho_b^h = \rho_b|_{\theta^b = f^b(y, \theta_-)},\ \rho_s^h = \rho_s|_{\theta^s = f^s(y, \theta_-)}.$$

We define the notion of recursive equilibrium for an economy $\mathcal{E}$ with portfolio penalties.

Definition 2 *A recursive equilibrium for an economy $\mathcal{E}$ is a pair of functions $(f, g) \in W$ satisfying*

$$F(f, g) \equiv 0.$$

The main reason for our interest in a recursive equilibrium is that we want a endogenous state space of small dimension. However, Kubler and Schmedders (2002) show that recursive equilibria do not always exist. The non-existence result of Krebs (2001) is not relevant for our purposes since we include portfolio constraints.

The following argument justifies our assumption of a recursive equilibrium. Suppose, we have a computational procedure that finds continuous approximations $(\hat{f}, \hat{g})$ to the policy and price function satisfying

$$\sup_{\theta \in \Theta} \hat{F}(\hat{f}, \hat{g})(\theta) < \epsilon$$

for some small error tolerance ϵ, where $\hat{F}(\hat{f},\hat{g})$ denotes the relative error in the functional equations, that is each equation in $F(\hat{f},\hat{g})$ divided by the current-period utility derivative term (for the first equation, for example, $\hat{F}^y_{1b}(\hat{f},\hat{g})(\theta_-) = F^y_{1b}(\hat{f},\hat{g})(\theta_-)/((g^b(y,\theta_-)+\omega^f_b)u'_1(c^1(y,\theta_-)))$.

In such a case we have clearly found an approximation to a recursive equilibrium; we could say, we have a "computational proof" of existence of a recursive equilibrium justifying our assumption to consider only such equilibria. In all the examples we have computed so far we always found a recursive equilibrium indicating that the class of economies for which a recursive equilibrium exists appears to be rather large.

As we will see below, it turns out that wealth is in a one-to-one equilibrium relationship with the state variables, that is, the portfolio holdings of one agent. Along the equilibrium path, it never happens that two different portfolio holding, price pairs occur which yield the same wealth. A possible interpretation is that for each given wealth level there is a unique continuation for the economy because equilibrium is globally unique. These results merit further research since using wealth as a state variable would reduce the endogenous state space and allow for arbitrarily many assets, see Kubler and Schmedders (2001).

2.6 Bounding the Endogenous State Space

Both to prove existence of an ergodic equilibrium and to approximate such an equilibrium it is important to ensure that the set of portfolios that can be part of an optimal solution for the agents is bounded; Duffie et al. (1994) discuss the role of compactness for existence; for computations boundedness is important because we compute equilibrium prices and portfolios as a function of some state variables and we want to approximate this function with splines with a finite number of nodes. Obviously, we cannot expect to approximate the equilibrium price and portfolio functions well on the endogenous state space Θ if this set is unbounded.

As mentioned above, from a theoretical point of view, there are a variety of possibilities to rule out Ponzi-Schemes - the most commonly used being implicit or explicit debt constraints. However, unlike in the case of a single asset (see Judd et al. (1999)), a debt constraint does not imply restrictions on the norm of agents' portfolios when there are two assets. An agent can go arbitrarily short in one asset if he covers that position with sufficient holdings of the other asset. For example, consider the case where agents are not allowed to hold negative wealth, that is, $e + \theta^s(q+d) + \theta^b > 0$ has to hold at all times. Then for any stock position $\theta^s \in \mathbb{R}$ it is possible to find a sufficiently large θ^b such that the debt constraint is satisfied. For this reason we have to introduce portfolio restrictions in order to obtain a bounded region for which we need to evaluate the policy and price functions. The short-sale constraints immediately result in a compact feasible portfolio set for the agents. The portfolio penalties result in endogenous bounds on possibly optimal portfolio choices which in turn also allow us to restrict our computations to a compact set. We emphasize that both constraints are more than just a technical or computational artifact helping with the analysis. Indeed, as our simulations in Section 4 show, the short-sale constraints are often binding; similarly, the agents frequently hold portfolios for which they receive a utility penalty. Henceforth, each of these constraints does influence the nature of the economic equilibrium. Most likely, the nature will be altered in comparison to the equilibrium of an economy without any portfolio restriction. It is important to keep these facts in mind when analyzing the computational results. Zhang (1997 a,b) provides a thorough analysis of the effect of different specifications of debt and short-sale constraints for a model with a single asset and shows that the specifications affect the results considerably.

In summary, the impossibility to state reasonable exogenous bounds on the space of agents' portfolios represents a serious problem in the computation of stationary equilibria. It appears to us that there is no simple approach to circumvent this problem.

3 A Spline Collocation Algorithm

In order to examine the equilibrium behavior of our model we need to know the portfolio policy functions $f = ((f_1^b, f_1^s), (f_2^b, f_2^s), \ldots, (f_S^b, f_S^s))$ and price functions $g = ((g_1^b, g_1^s), (g_2^b, g_2^s), \ldots, (g_S^b, g_S^s))$. Since we cannot compute closed-form expressions for these functions, we describe a new method for computing very good approximations of these functions.

3.1 Representation of the Approximating Functions

We approximate $f(y, \theta^b, \theta^s) = (f^b(y, \theta^b, \theta^s), f^s(y, \theta^b, \theta^s))$ parametrically by functions

$$\hat{f}^b(y, \theta^b, \theta^s) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^{by} B_i(\theta^b) B_j(\theta^s)$$

and

$$\hat{f}^s(y, \theta^b, \theta^s) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^{sy} B_i(\theta^b) B_j(\theta^s)$$

and $g(y, \theta^b, \theta^s) = (g^b(y, \theta^b, \theta^s), g^s(y, \theta^b, \theta^s))$ by

$$\hat{g}^b(y, \theta^b, \theta^s) = \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij}^{by} B_i(\theta^b) B_j(\theta^s)$$

and

$$\hat{g}^s(y, \theta^b, \theta^s) = \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij}^{sy} B_i(\theta^b) B_j(\theta^s)$$

where n^2 is the number of terms used. This tensor-product approach to two-dimensional approximation is reasonable because the dimension is low and the approach constitutes a straightforward extension of the one-dimensional techniques (see Judd et al. (1999)) to higher dimensions. The functions B_k are B-splines of order 4. These functions yield a linearly independent basis for one-dimensional cubic splines. For our approximations we use cubic splines instead of orthogonal polynomials since our price functions sometimes exhibit high curvature, in particular as the penalty function gets large. As Judd (1992) points out, orthogonal polynomials fair poorly in approximating functions with high curvature. On the contrary, the cubic splines produced very small errors in all regions of the domain.

Given a grid of knots (x_i) order k B-splines are recursively defined by:

$$B_i^k(x) = \frac{x - x_i}{x_{i+k} - x_i} B_i^{k-1} + \frac{x_{i+k+1} - x}{x_{i+k+1} - x_{i+1}} B_{i+1}^{k-1}(x)$$

with

$$B_i^0(x) = \begin{cases} 0 & x < x_i \\ 1 & x_i \leq x \leq x_{i-1} \\ 0 & x \geq x_i \end{cases}$$

For an overview about B-splines see Judd (1998). De Boor (1978) gives a detailed introduction into the representation of B-splines and how they can be applied to the approximation of functions. Moreover, he provides numerous subroutines that we use for our computations.

3.2 Collocation

Given an income state y, approximating functions $\hat{f}$ and $\hat{g}$ and agent 1's beginning-of-period portfolio holding θ we define the following notation.

$$\begin{aligned} c^1 &= e^1(y) + \theta^b + \theta^s(\hat{g}^s(y,\theta) + d(y)) - \hat{f}(y,\theta)\hat{g}(y,\theta) - \omega(\theta, \hat{f}(y,\theta)) \quad (7) \\ \tilde{c}^1_+ &= e^1(\tilde{y}) + \hat{f}^b(y,\theta) + \hat{f}^s(y,\theta)\left[\hat{g}^s(\tilde{y}, \hat{f}(\tilde{y},\theta)) + d(\tilde{y})\right] - \\ & \quad \hat{g}(\tilde{y}, \hat{f}(y,\theta))\hat{f}(\tilde{y}, \hat{f}(y,\theta)) - \omega(\hat{f}(y,\theta), \hat{f}(\tilde{y}, \hat{f}(y,\theta))) \\ c^2 &= e(y) - c^1 - 2\omega(\theta, \hat{f}(y,\theta)) \\ \tilde{c}^2_+ &= e(\tilde{y}) - \tilde{c}^1_+ - 2\omega(\hat{f}(y,\theta), \hat{f}(\tilde{y}, f(y,\theta))) \end{aligned}$$

where c^i denotes the approximation for agent i's current period consumption, and $\tilde{c}^i_+$ denotes the approximation for agent i's random next period consumption. Moreover, $\omega_b^{\hat{f}}, \tilde{\omega}_{s1}^{\hat{f}}, \omega_{s2}^{\hat{f}}, \rho_b^{h,\hat{f}}, \rho_s^{h,\hat{f}}$ are defined by substituting $\hat{f}$ for f into the corresponding terms. Substituting our approximations into the functional equations $F(f,g) = 0$ results into the following system of equations.

$$(\hat{g}^b(y,\theta) + \omega_b^{\hat{f}})u_1'(c^1) - \rho_b^{h,\hat{f}} - \beta_1 E_y(u_1'(\tilde{c}^1_+)) = 0 \quad (8)$$

$$(\hat{g}^s(y,\theta) + \omega_s^{\hat{f}})u_1'(c^1) - \rho_s^{h,\hat{f}} - \beta_1 E_y\left\{(\hat{g}^s(\tilde{y}, f(y,\theta)) + d(\tilde{y}) - \omega_{s1}^{\hat{f}})u_1'(\tilde{c}^1_+)\right\} = 0 \quad (9)$$

$$(\hat{g}^b(y,\theta) + \omega_b^{\hat{f}})u_2'(c^2) - \rho_b^{h,\hat{f}} - \beta_2 E_y(u_2'(\tilde{c}^2_+)) = 0 \quad (10)$$

$$(\hat{g}^s(y,\theta) + \omega_s^{\hat{f}})u_2'(c^2) - \rho_s^{h,\hat{f}} - \beta_2 E_y\left\{(\hat{g}^s(\tilde{y}, f(y,\theta)) + d(\tilde{y}) - \omega_{s1}^{\hat{f}})u_2'(\tilde{c}^2_+)\right\} = 0 \quad (11)$$

To compute the coefficients $(a_{ij}^{by}), (a_{ij}^{sy}), (b_{ij}^{by})$ and (b_{ij}^{sy}) we select a collocation grid of as many mesh-points $(\theta_i^b, \theta_j^s)_{i,j=1,\ldots,n,}$ as we have unknown coefficients for the approximating functions. We take these points to be equal to θ and obtain 4 equations for each point and each income state from (1). Notice that the problem has been transformed from finding functions f and g solving the Euler equations over the continuous state space to finding a zero of a system of $4 \cdot S \cdot n^2$ nonlinear equations that has the real coefficients $a = ((a_{ij}^{by}), (a_{ij}^{sy}))$ and $b = ((b_{ij}^{by}), (b_{ij}^{sy}))$ as unknowns.

Two theoretical questions of central importance are whether or under what conditions the system possesses a solution, and if the functions $\hat{f}$ and $\hat{g}$ converge to the true policy functions f and g as the number of collocation points tends to infinity. We do not examine this problem here, the interested reader should consult Zeidler (1986) or Krasnosel'skii and Zabreiko (1984) for related problems.

3.3 Time Iteration

The described system of equations can be very large. With $S = 8$ exogenous income states and a grid of 15 by 15 collocation points, as in the example in Section 5, the system has 7200 equations and unknowns. While there is some sparsity in this system due to the structure of B-splines it is difficult to exploit it. To solve the entire system with Newton-style algorithms is extremely difficult due to the enormous amount of memory needed and because the system is rather ill-conditioned. Iterative Gauss-Seidel methods (see Ortega and Rheinboldt (1970)) are unlikely to work since the system lacks the desirable diagonal dominance structure; furthermore, Gauss-Seidel methods would require costly recomputation of the B-spline coefficients after solving each specific equation.

We instead follow a simple Gauss-Jacobi approach which can be motivated by the following economic intuition. It seems reasonable to hope that finite-horizon models with a very large number of time periods are a good approximation to the infinite-horizon model under consideration, although this fact has not been proven rigorously. A natural approach for computing portfolio holdings and prices in a finite-horizon model is backward induction. Therefore we compute the approximate policy functions $\hat{f}$ and $\hat{g}$ through an iterative process starting with some initial guess f_0 and g_0. In each iteration $k = 1, 2, \ldots$, the algorithm solves the Euler equations for all collocation points $\theta = (\theta^b, \theta^s) \in G$ and income states $y \in Y$ by computing current portfolio decisions θ and corresponding asset price q given functions $\hat{f}_{k-1}$ and $\hat{g}_{k-1}$ governing the policy process in the subsequent period. The coefficients a and b for the new functions $\hat{f}_k$ and $\hat{g}_k$, respectively, are then determined through interpolation. The algorithm terminates if

$$\max_{(\theta^b,\theta^s)\in G, y\in Y}\{|\hat{f}_k(y,\theta) - \hat{f}_{k-1}(y,\theta)|, |\hat{g}_k(y,\theta) - \hat{g}_{k-1}(y,\theta)|\} < \epsilon.$$

3.4 Solving the Euler Equations

During each iteration for given functions $\hat{f}_{k-1}$ and $\hat{g}_{k-1}$ it is necessary to solve the system of 4 Euler equations at all collocation points $\theta = (\theta^b, \theta^s) \in G$ for each income state $y \in Y$. Newton-type algorithms regularly fail to solve these systems because they are highly nonlinear and ill-conditioned for many collocation points. The key insight for solving these systems is that they are similar to the equilibrium conditions of the well-known General Equilibrium Model with Incomplete Asset Markets (GEI Model). Therefore, in order to solve the Euler equations we can apply - with some modifications - the homotopy algorithm developed by Schmedders (1998) for the GEI Model. For a general description of the homotopy principle see Garcia and Zangwill (1981).

We denote the homotopy parameter by λ and the collocation point under consideration by $\theta_- = (\theta^b_-, \theta^s_-)$. Our 4 homotopy equations are now defined as follows:

$$\begin{aligned}
0 &= (q^b+\omega_b)u_1'(e^1(y)+\theta^b_- + \theta^s_-(q^s+d(y)) - \theta q - \omega(\theta_-,\theta)) - \rho^h_b - \\
&\quad \beta_1 E_y[u_1'(e^1(\tilde{y}) + \theta^b + \theta^s\left[\hat{g}^s_{k-1}(\tilde{y},\theta) + d(\tilde{y})\right] - \hat{g}_{k-1}(\tilde{y},\theta)\hat{f}_{k-1}(\tilde{y},\theta) - \omega(\theta,\hat{f}_{k-1}(\tilde{y},\theta)))] \quad (12)\\
0 &= (q^s+\omega_s)u_1'(e^1(y)+\theta^b_- + \theta^s_-(q^s+d(y)) - \theta q - \omega(\theta_-,\theta)) - \rho^h_s - \\
&\quad \beta_1 E_y[(\hat{g}^s_{k-1}(\tilde{y},\theta) + d(\tilde{y}) - \omega_{s1})u_1'(e^1(\tilde{y}) + \theta^b + \theta^s\left[\hat{g}^s_{k-1}(\tilde{y},\theta) + d(\tilde{y})\right] - \\
&\quad \hat{g}_{k-1}(\tilde{y},\theta)\hat{f}_{k-1}(\tilde{y},\theta) - \omega(\theta,\hat{f}_{k-1}(\tilde{y},\theta)))] \quad (13)\\
0 &= \lambda[(q^b+\omega_b)u_2'(e^2(y) - \theta^b_- - \theta^s_-(q+d(y)) + \theta q - \omega(\theta_-,\theta)) - \rho^h_b -
\end{aligned}$$

$$
\begin{aligned}
& \beta_2 E_y[u_2'(e^2(\tilde{y}) - \theta^b - \theta^s \left[\hat{g}^s_{k-1}(\tilde{y},\theta) + d(\tilde{y})\right] + \hat{g}_{k-1}(\tilde{y},\theta)\hat{f}_{k-1}(\tilde{y},\theta) - \omega(\theta, \hat{f}_{k-1}(\tilde{y},\theta)))]] \\
& -(1-\lambda)(\theta^b_- - \theta^b) \qquad (14) \\
0 \quad = \quad & \lambda[(q^s + \omega_s)u_2'(e^2(y) - \theta^b_- - \theta^s_-(q^s + d(y)) + \theta q - \omega(\theta_-, \theta)) - \rho^h_s - \\
& \beta_2 E_y[(\hat{g}^s_{k-1}(\tilde{y},\theta) + d(\tilde{y}) - \omega_{s1})u_2'(e^2(\tilde{y}) - \theta^b - \theta^s \left[\hat{g}^s_{k-1}(\tilde{y},\theta) + d(\tilde{y})\right] + \\
& \hat{g}_{k-1}(\tilde{y},\theta)\hat{f}_{k-1}(\tilde{y},\theta) - \omega(\theta, \hat{f}_{k-1}(\tilde{y},\theta)))]] - (1-\lambda)(\theta^s_- - \theta^s) \qquad (15)
\end{aligned}
$$

Note that the first two equations do not include the homotopy parameter. Only the Euler equations of agent 2 are perturbed. For $\lambda = 0$ these two equations have a trivial solution, namely $\theta^b = \theta^b_-$ and $\theta^s = \theta^s_-$, that is, there is no trade on the financial markets. Substituting these values in the Euler equations of the first agent lead to two remaining equations with the two unknowns q^b and q^s. This small system can easily be solved, both in closed form and numerically via Newton's method; it has a unique solution, which is the starting point of a homotopy path. Although we cannot prove convergence of the homotopy, in practice, the path always leads to a solution of the four equations with $\lambda = 1$. This solution is the desired solution of the Euler equations at the given collocation point.

While the homotopy approach always finds a solution it can be very time-consuming. After a few iterations the difference between the old and new policy functions is usually fairly small. We can therefore take the old policy function as a starting point for a Newton method which then finds a solution to the Euler equations rather quickly. Whenever the Newton method fails to find a solution, the homotopy approach is used.

3.5 The Algorithm

Our algorithm can be summarized as follows:

Time Iteration Spline Collocation Algorithm

Step 0: Select an error tolerance ϵ for the stopping criterion, a set G of collocation points, and a starting point $\hat{f}_0, \hat{g}_0$.

Step 1: Given functions $\hat{f}_{k-1}, \hat{g}_{k-1}$, $\forall \theta \in G$, $\forall y \in Y$ compute $\theta^1(y,\theta)$ and $q(y,\theta)$ finding a zero of the residual functions.

Step 2: Compute the new approximations $\hat{f}_k, \hat{g}_k$, that is, the new coefficients a and b using interpolation and $\theta^1(y,\theta), q(y,\theta)$.

Step 3: Check stopping criterion: If $\max_{(\theta^b,\theta^s)\in G, y\in Y}\{|\hat{f}_k(y,\theta) - \hat{f}_{k-1}(y,\theta)|, |\hat{g}_k(y,\theta) - \hat{g}_{k-1}(y,\theta)|\} < \epsilon$. then go to Step 4. Otherwise increase k by 1 and go to Step 1.

Step 4: The algorithm terminates. Set $\hat{f} = \hat{f}_k, \hat{g} = \hat{g}_k$.

3.6 Important Aspects of the Algorithm

We concisely discuss a number of issues which are important for the implementation and application of our algorithm.

3.6.1 Collocation Grid

For a satisfactory approximation we chose a collocation grid of size 15 by 15; the results suggest that this is much finer than necessary. The reader should note that we use the collocation grid for collocation and interpolation purposes, and that we allow agents to hold any portfolio between the collocation points. In contrast, Heaton and Lucas (and several other papers) force agents to choose portfolios that always lie on the same grid. Their grid of 30 points per agent appears to be finer than ours but actually results in a far cruder approximation. In their model, an agent must trade an amount equal to the distance between two grid points; since there are only 30, this amount is about 3% of the range in permissible holdings.

3.6.2 Starting Points

The choice of the starting points $\hat{f}_0, \hat{g}_0$ significantly affects the running times. As is always the case, it is advantageous to have good starting points. When solving multiple examples, we typically use previously computed coefficients from earlier problems as starting points for new problems, in particular when we only slightly alter the model parameters during the sensitivity analysis. This approach to initial guesses produces substantial savings when doing many problems.

3.6.3 Shape Preservation

So far we have not addressed the possibility of shape problems in our calculations. Using cubic splines for the approximation of the equilibrium transition functions does not ensure that the shape of the true functions is preserved. For example, it could happen that the functional points at the collocation points exhibits a property such as monotonicity or convexity which is lost through the spline approximation. A safer approach for the approximations would therefore be a shape-preserving method (see Judd (1998)). However, looking at the Figure 2 in Section 4 we see, that such shape problems apparently did not arise in our calculations.

3.6.4 Boundary Condition for Splines

The economic model does not naturally impose a condition on the slope of the policy and price functions at the boundary of the state space. Therefore, we also do not want to impose a boundary condition on the spline approximation. Instead we use the not-a-knot condition in our implementation. The jump in the third derivative at the second collocation point from the boundary is forced to zero; thus, the first and the second cubic polynomial pieces are made to coincide.

3.6.5 Interpolation on a Bounded Set

The interpolation of the policy functions $\hat{f}(y, \cdot) : \Theta \to \Theta$ on the compact set Θ could lead to problems with the simulation of the model. A typical interpolation procedure does not take into account any bounds on the function being interpolated. For every grid point $\theta_- \in \Theta$ the first-order conditions ensure that $\hat{f}(y, \theta_-) \in \Theta$. However, the interpolation might result in the existence of points $\tilde{\theta} \in \Theta$ such that $\hat{f}(y, \tilde{\theta}) \notin \Theta$. In the model with short-sale constraints these constraints would be violated at such a point. In addition, such a function would lead to an error in the simulation of the model, when the function $\hat{f}$ is effectively composed with itself many thousand times. Since the interpolation is only valid for the set Θ, an evaluation of $\hat{f}$ at a point $\hat{f}(y, \tilde{\theta}) \notin \Theta$ could generate huge simulation errors.

We have always checked our simulations for the occurrence of function values outside the set Θ. While such values are theoretically possible, we have so far never detected them in any of our examples. In particular in the models with utility penalties these penalties result in a very strong inward-pointing form of the policy functions once the penalty becomes sufficiently large.

3.6.6 Implementation in FORTRAN

We implemented our algorithm in FORTRAN on a PENTIUM-233 computer. For the homotopy path-following in Step 1 of the algorithm we used the software package HOMPACK, a suite of FORTRAN 77 subroutines for solving systems of nonlinear equations using homotopy methods (Watson, Billups, and Morgan (1987)). The necessary computations involving the B-splines were done with the FORTRAN subroutines provided in deBoor (1978). For the Newton-solver we used HYBRD, a suite of FORTRAN subroutines for solving nonlinear equations using Powell's hybrid method.

3.6.7 Euler Equations with Short-Sale Constraints

So far we have described our algorithm for economies with portfolio penalties. In order to apply it to the computation of equilibria with short-sale constraints we have to make only one significant adjustment. With short-sale constraints the agents face utility optimization problems with inequality constraints resulting in first-order conditions of optimality which include Lagrange multipliers and inequalities. Denoting by μ_t^{hb} and μ_t^{hs} the Lagrange multipliers for the short-sale constraint on the bond and stock, respectively, at time t we obtain the following first-order conditions for agent h :

$$\begin{aligned}
(q_t^b + \omega_b) u_h'(c_t) &= \beta_h E_t(u_h'(c_{t+1}))) + \mu_t^{hb} \\
\mu_t^{hb}(\theta_t^b + B^{hb}) &= 0 \\
\mu_t^{hb} &\geq 0 \\
\theta_t^b + B^{hb} &\geq 0 \\
(q_t^s + \omega_{s2}) u_h'(c_t) &= \beta_h E_t \left\{ (q_{t+1}^s + d_{t+1} - \omega_{s1}) u_h'(c_{t+1}) \right\} + \mu_t^{hs} \\
\mu_t^{hs}(\theta_t^s + B^{hs}) &= 0 \\
\mu_t^{hs} &\geq 0 \\
\theta_t^s + B^{hs} &\geq 0
\end{aligned}$$

Due to the inequalities the first-order conditions cannot be simply solved by a nonlinear-equations algorithm such as Newton's method or the homotopy method we are using. However, through a simple trick (see Garcia and Zangwill (1981)) we can eliminate all inequalities and state the optimality conditions as a system consisting solely of equations.

Let l be a natural number and $\eta^{ha} \in \mathbb{R}$ for $h = 1, 2$ and $a \in \{b, s\}$. Note the following relations:

$$(\max\{0, \eta^{ha}\})^l = \begin{cases} (\eta^{ha})^l & \text{if } \eta^{ha} > 0 \\ 0 & \text{if } \eta^{ha} \leq 0 \end{cases}$$

$$(\max\{0, -\eta^{ha}\})^l = \begin{cases} 0 & \text{if } \eta^{ha} > 0 \\ |\eta^{ha}|^l & \text{if } \eta^{ha} \leq 0 \end{cases}$$

Moreover,

$$(\max\{0, \eta^{ha}\})^l \cdot (\max\{0, -\eta^{ha}\})^l \geq 0, \text{ and } (\max\{0, \eta^{ha}\})^l \cdot (\max\{0, \eta^{ha}\})^l = 0.$$

We define

$$\mu^{ha} = (\max\{0, \eta^{ha}\})^l \text{ and } \theta_t^a + B^{ha} = (\max\{0, -\eta^{ha}\})^l$$

which allows us to state first-order conditions of optimality as a system of equations which is equivalent to the system above:

$$\begin{aligned} (q_t^b + \omega_b)u_h'(c_t) &= \beta_h E_t(u_h'(c_{t+1}))) + (\max\{0, \eta^{hb}\})^l \\ \theta_t^b &= -B^{hb} + (\max\{0, -\eta^{hb}\})^l \\ (q_t^s + \omega_{s2}u_h'(c_t) &= \beta_h E_t \left\{ (q_{t+1}^s + d_{t+1} - \omega_{s1})u_h'(c_{t+1}) \right\} + (\max\{0, \eta^{hs}\})^l \\ \theta_t^s &= -B^{hs} + (\max\{0, -\eta^{hs}\})^l \end{aligned}$$

We have transformed the system of optimality conditions into a system of 4 equations for every agent with the variables θ^h and $\eta^h = (\eta^{hb}, \eta^{hs})$. At every collocation point we have thus to solve a system of 8 equations in the unknowns θ, q, η^1 and η^2. By a proper choice of the number l we can obtain any desired degree of differentiability of the functions in our equations. Typically we used $l = 3$ or $l = 4$ so that the function $(\max\{0, \eta^{hs}\})^l$ was twice or three times continuously differentiable, respectively.

Note that the reason why we can add short-sale constraints with relatively little effort lies in the fact that we use a time-iteration algorithm. If we wanted to use a Newton-Method to solve the collocation equations directly this would double the number of unknowns.

4 Example

In this section we consider an example of an exchange economy roughly calibrated to yearly US data. We illustrate the various conceptual difficulties in computing equilibria. We show the behavior of our algorithm and report running times and maximum errors for all the computations.

There are two agents with constant relative risk aversion preferences of the form $u^h(c) = \frac{c^{1-\gamma_h}}{1-\gamma_h}$, and a discount factor of $\beta = 0.96$. There are eight exogenous income states, the individual labor incomes and the stock's dividends are given by

$$e^1 = \begin{pmatrix} 3.15 \\ 3.15 \\ 7.35 \\ 7.35 \\ 4 \\ 4 \\ 6 \\ 6 \end{pmatrix}, e^2 = \begin{pmatrix} 7.35 \\ 7.35 \\ 3.15 \\ 3.15 \\ 6 \\ 6 \\ 4 \\ 4 \end{pmatrix}, \text{ and } d = \begin{pmatrix} 1.4 \\ 1.8 \\ 1.4 \\ 1.8 \\ 1.4 \\ 1.8 \\ 1.4 \\ 1.8 \end{pmatrix}.$$

Individual endowments in the stock are given by $\theta^{1s}_{-1} = \theta^{2s}_{-1} = 0.5$. The Markov transition matrix is given by

$$\mathbf{P} = \begin{pmatrix} 0.3 & 0.2 & 0.125 & 0.125 & 0.075 & 0.075 & 0.005 & 0.005 \\ 0.2 & 0.3 & 0.125 & 0.125 & 0.075 & 0.075 & 0.005 & 0.005 \\ 0.125 & 0.125 & 0.3 & 0.2 & 0.05 & 0.05 & 0.075 & 0.075 \\ 0.125 & 0.125 & 0.2 & 0.3 & 0.05 & 0.05 & 0.075 & 0.075 \\ 0.075 & 0.075 & 0.05 & 0.05 & 0.3 & 0.2 & 0.125 & 0.125 \\ 0.075 & 0.075 & 0.05 & 0.05 & 0.2 & 0.3 & 0.125 & 0.125 \\ 0.005 & 0.005 & 0.075 & 0.075 & 0.125 & 0.125 & 0.3 & 0.2 \\ 0.005 & 0.005 & 0.075 & 0.075 & 0.125 & 0.125 & 0.2 & 0.2 \end{pmatrix}.$$

The persistence of the income distribution process is comparable to the process in Heaton and Lucas (1996) (of course, Heaton and Lucas consider a growing economy while we limit ourselves to an economy which is stationary in endowments).

To fix ideas we start with the case where there are no transaction costs (i.e. $\tau^s = \tau^b = 0$) and where the short sale constraints are given by $\alpha^b_1 = -2.5$, $\alpha^s_1 = 0$, $\alpha^b_2 = 2.5$ and $\alpha^s_2 = 0$. That is we do not allow short sales of the stock but agents are allowed to borrow a substantial fraction of their yearly income when poor.

[FIGURE 1 ABOUT HERE]

Identical Preferences

We first assume that both agents have identical relative risk aversion of $\gamma = 1.5$. As explained in Section 3 above we calculate the equilibrium policy functions as functions of portfolio holdings. Figure 1 shows the pricing function for the stock price for state 1. In order to obtain a symmetric scale we always refer to agent 1's net (actual portfolio minus initial portfolio) asset holdings as equilibrium portfolios. The figure tells us for every possible portfolio of agent 1 in period $t-1$ the stock price in period t, but it does not tell us which of these prices actually occur on the equilibrium path.

Figure 1 is not very informative, because one needs to know which portfolio holdings occur along a given equilibrium path. In order to investigate the equilibrium behavior of the model we simulate the economy for 20000 periods (We actually simulate for 21000 periods and only use the last 20000 periods in order to eliminate any effects which might be caused by initial conditions.).

Figure 2 shows a scatter plot of all portfolio holdings which occur along the equilibrium path.

[FIGURE 2 ABOUT HERE]

The result is very surprising: There is only a one-dimensional subset of equilibrium portfolio holdings; we refer to this set as the 'invariant' set. Note that the 8 diagonal lines in the figure can be associated with the 8 different states. Figure 2 suggests that for models without trading frictions there exists a one-to-one equilibrium relationship between the wealth of an agent and his portfolio holdings.

Note that although we chose the short-sale constraints on the bond to be very weak they are binding for most of the time. When they are not binding equilibrium stock holdings are very small and they only increase as they hit the short-sale constraint.

In order to investigate the relationship between wealth and portfolio holdings we plot the resulting equilibrium wealth levels against portfolio holdings and prices for each state. For the purposes of this section we define wealth as agent 1's wealth net of initial asset holdings i.e. wealth in period t is defined as $e^1(y_t) + \theta^{1b}_{t-1} + (\theta^{1s}_{t-1} - \theta^{1s}_{-1})(q^s_t + d(y_t))$. This has the advantage that the economy's total wealth always sums up to aggregate endowments and is therefore independent of the wealth distribution (the wealth distribution for a model with H agents is thus $(H-1)$ -dimensional).

Without frictions the wealth level is in a one-to-one equilibrium relationship with the portfolio holdings (Kubler and Schmedders, 2001, provide a theoretical foundation for using the agents' wealth distribution as the only endogenous state variable for stationary equilibria and develop an algorithm for computing such equilibria.). In all subsequent figures we will plot the computed policy functions against agent 1's wealth level since it provides a good way to illustrate our results. Note that we plot actual simulated values along the equilibrium path. The only reason this is possible is that our results are very accurate so that even after 21000 periods the simulated equilibrium portfolio holdings still almost lie in the one-dimensional invariant set. Using wealth as the state variable has the huge advantage that it allows us to plot the equilibrium values as a function of a single variable. While it can be rather hard to interpret three-dimensional surface plots of the equilibrium functions a two-dimensional graph is easy to interpret. We will therefore report all transition functions as functions from wealth, although we have actually computed them as functions from portfolio holdings.

[FIGURES 3 – 6 ABOUT HERE]

Figure 3 shows the equilibrium portfolio holdings in state 1 associated with different wealth levels (the figures for the other states look very similar and are therefore not shown). The function which is increasing for large wealth levels denotes the stock holding - the stock holding only decreases for the region where the extended short-sale constraints are not binding. Note that the short-sale constraints are binding for most of the time. The two 'kinks' in the equilibrium functions are caused by the fact that one agent reaches a binding constraint at that point and is forced to hold fewer bonds for more stocks. As discussed in Section 2 above this is an unpleasant feature of the model and there is no easy solution to this problem, if one wants to use portfolio holdings as a state variable. As we show below this is also necessary if one wants to investigate an economy with transaction costs. Figure 4 depicts a histogram of the wealth levels from a simulation over 20000 periods; thus, 3000 occurrences of a particular wealth level correspond to a occurrence 15 percent of the time. Figures 5 and 6 show the equilibrium bond and stock price in state 1 as a function of agent's 1 wealth levels. Note that these figures – in conjunction with the histogram – prove that there is substantial variation in equilibrium prices. In state 1 agent 1 is rather poor – as he agent one has to borrow more and more money, agent 2 accumulates more wealth his precautionary demand for savings (assets) increases, and the asset return decreases.

Heterogeneous Risk-aversion

We now compute an equilibrium with coefficients of relative risk-aversion of $\gamma_1 = 2.5$ and $\gamma_2 = 0.5$ Figures 7–10 are the analogues of Figures 3–6 for this case.

[FIGURES 7 – 10 ABOUT HERE]

The average wealth level of the more risk-averse agent is slightly higher than in the case with identical preferences. As his wealth level increases his demand for assets increases hugely and the price (of all assets) rises.

Heterogeneous Discount Rates

With complete markets it is not interesting to consider examples where agents' differ with respect to their discount rates because the agent with the higher β will eventually hold all the wealth and asymptotically the economy will behave like an economy which is populated only by agent with high β 's. With incomplete markets however, this is usually not the case. We return to the case of identical risk-aversion but we assume now that $\beta_1 = 0.955$ and $\beta_2 = 0.96$.

Figures 11–14 show the analogues of Figures 7–10 for this case. The wealth-constraint (i.e. the short-sale constraint on the stock) is usually not binding for the impatient agent. Even though he is in debt most of the time his average debt level is significantly above his effective debt-constraint. The variation in equilibrium prices for the stock is substantial.

[FIGURES 11 – 14 ABOUT HERE]

Portfolio Penalties

For more realistic models it is much easier from a numerical point of view to restrict agents' portfolio holdings not by short-sale constraints but by penalizing portfolio holdings as explained in Section 2 above. We return to the case of identical preferences and – instead of assuming short-sale constraints – we assume a penalty of the form $\kappa^s = 0$, $\kappa^b = 1.0$ and $L^{hb} = -2.25$. That is the agents start experiencing a utility penalty if their bond-holding goes below -2.25 Figures 15–18 show the results.

[FIGURES 15 – 18 ABOUT HERE]

For most wealth levels the penalty function approach seems to work very well. The actual bond-holdings almost always lie above -2.85. Note, however, that at the points where the portfolio penalty sets in their is substantial inaccuracy in our computations.

If one wants to use the penalty approach to restrict agents' portfolios one has to be well aware of the fact that one does compute a slightly different equilibrium. Note, however, that both pricing functions are almost identical to the pricing functions in Figures 3 and 4 and that the histogram of possible wealth levels is also very similar to the histogram in case of short-sale constraints. Overall, taking into account the running-times and maximum errors which we report in the Appendix, it seems like portfolio-penalties can be a useful alternative to short-sale constraints.

Transaction costs

We now examine how positive transaction costs on stock-trading affect the equilibrium outcome. We consider the basic case of identical preferences and short-sale constraints from above and assume that there are substantial transaction costs on trading the stock, $\tau_s = 0.5$ (this number seems high at first but one has to take into account that the entire stock represents the total productive wealth of the economy and assuming that trading this costs 5% of aggregate consumption might not even be too unrealistic). Figures 19–22 show the results.

[FIGURES 19 – 22 ABOUT HERE]

The figures show that wealth is not a sufficient statics for portfolio-holdings and prices anymore. Somewhat surprisingly, the savings-behaviour (i.e. bond-holding) is completely different with transaction costs while the stock-holdings are only affected slightly.

Patient Agents

To demonstrate the ability of our algorithm we also compute equilibria for the case of identical tastes but where each agents' discount factor is $\beta^h = 0.99$ – this roughly corresponds to quarterly data (note however that we leave the calibration for income states unchanged). From an economic point of view the results are not overly interesting. The pricing functions are very similar to the pricing function for $\beta = 0.96$ - this is not very surprising because tastes are identical. However, the fact that we can compute economies calibrated to quarterly data offers possibilities for further research.

4.1 Errors and Running Times

In order to evaluate the quality of our approximations to the true equilibrium transition functions, we compute the residuals of the Euler equations. In order to obtain relative errors we divide them by the price times the agents current period marginal utility. We evaluate these errors at 100 times 100 points in our state space and report the maximum error, the average errors usually lie around one to two orders of magnitude below these maximum errors. This way of reporting errors is similar to the errors reported in Heaton and Lucas (1996). For each agent's Euler equation they compute the price which makes it hold exactly and then take the difference of these prices.

The reported running times always refer to running times with $f = g = 0$ as a starting point. Running times can be improved with better starting points.

Errors and running times for Section 4.1

For relatively low beta and a model with only few states the running times appear very reasonable.

Table 3: Running Times and Errors

	Maximum error in %	Running Time in hours.minutes
identical	$1.01 \cdot 10^{-5}$	2.16
different γ	$9.89 \cdot 10^{-5}$	3.29
different β	$7.29 \cdot 10^{-6}$	2.02
trans. cost	$5.34 \cdot 10^{-6}$	1.54
penalties	$3.85 \cdot 10^{-5}$	2.02
$\beta = 0.99$	$5.21 \cdot 10^{-4}$	8.12

5 Conclusion

This paper develops a computational strategy for incomplete asset market analysis which allows agents to hold any portfolio, and which allows them to trade more frequently than is allowed by earlier methods. We also allow for various forms of portfolio constraints, transaction costs, and portfolio penalties.

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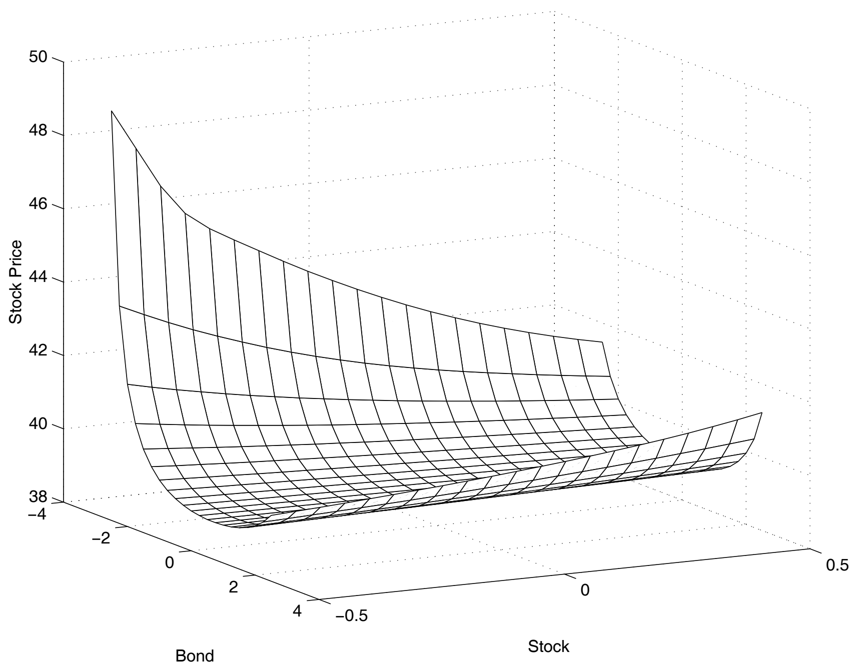


Figure 1. Equilibrium pricing function

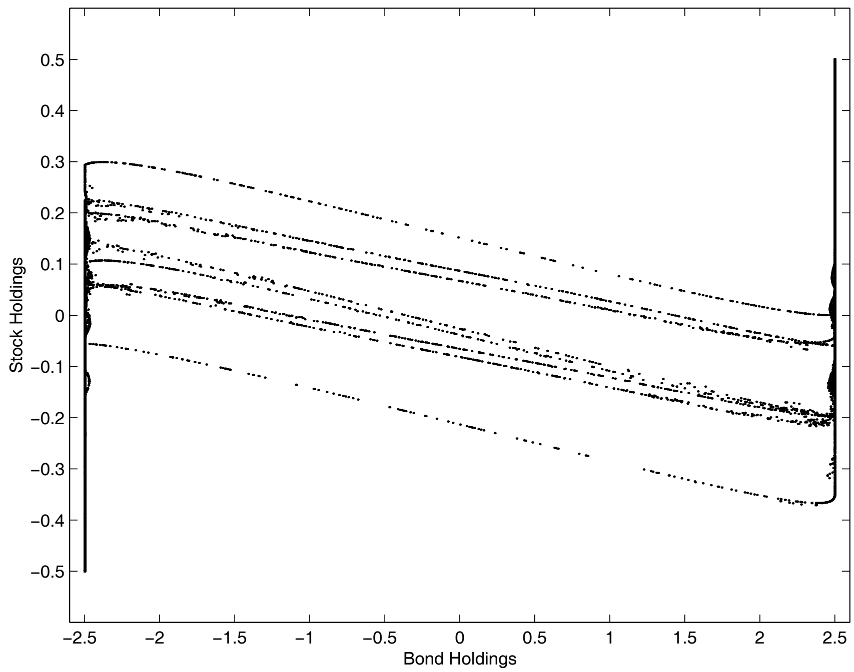


Figure 2. Scatter plot of equilibrium portfolios

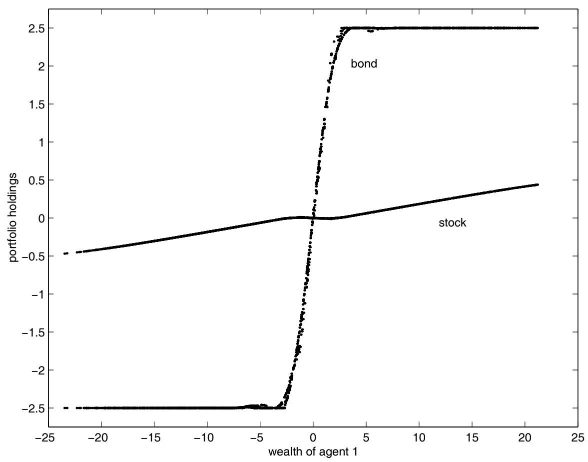


Figure 3. Portfolio holdings

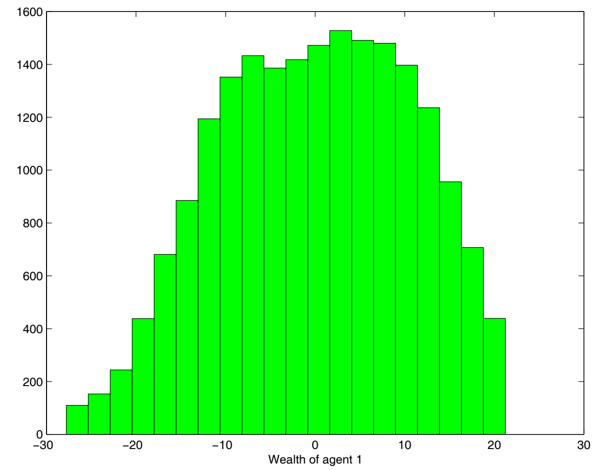


Figure 4. Histogram of wealth levels

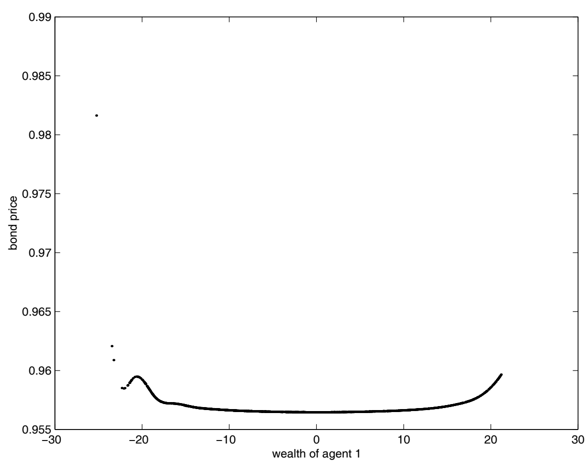


Figure 5. Bond price

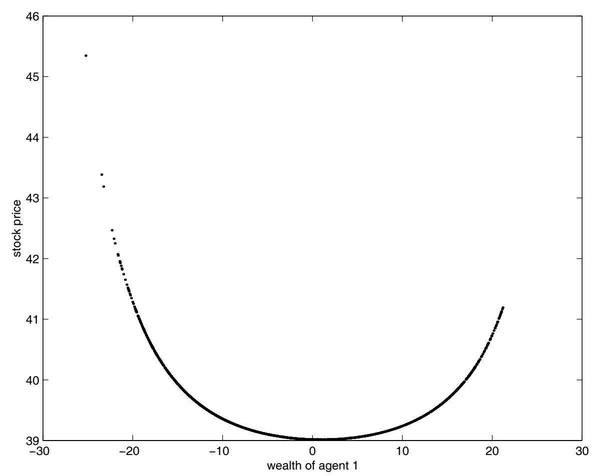


Figure 6. Stock price

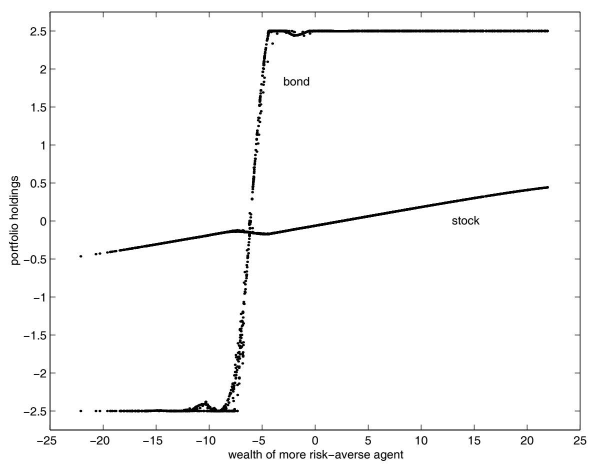


Figure 7. Portfolio holdings

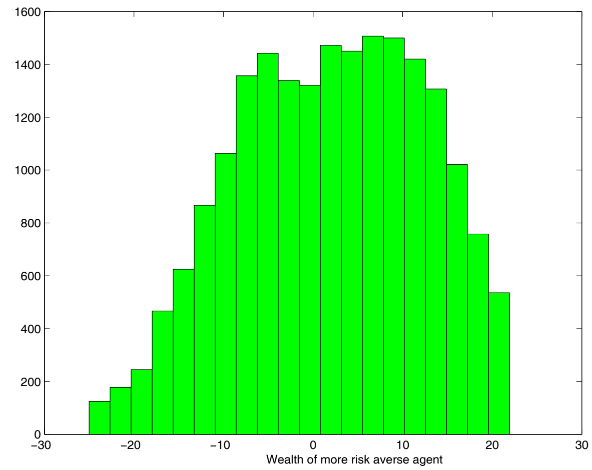


Figure 8. Histogram of wealth levels

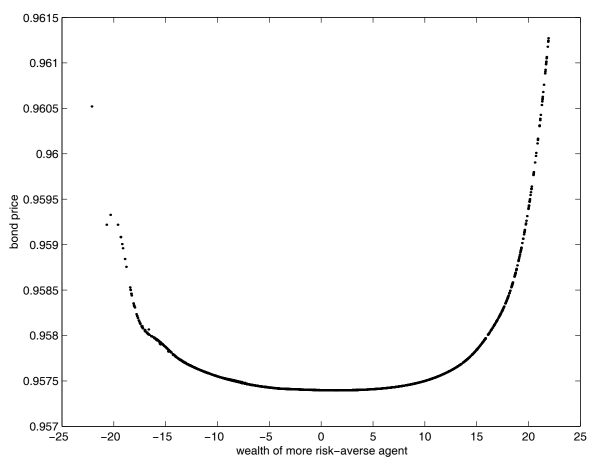


Figure 9. Bond price

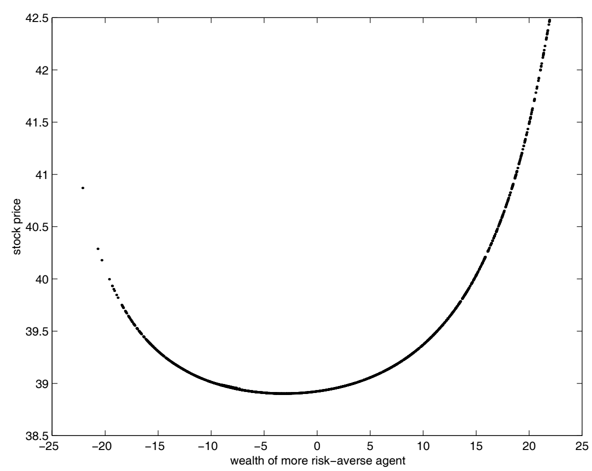


Figure 10. Stock price

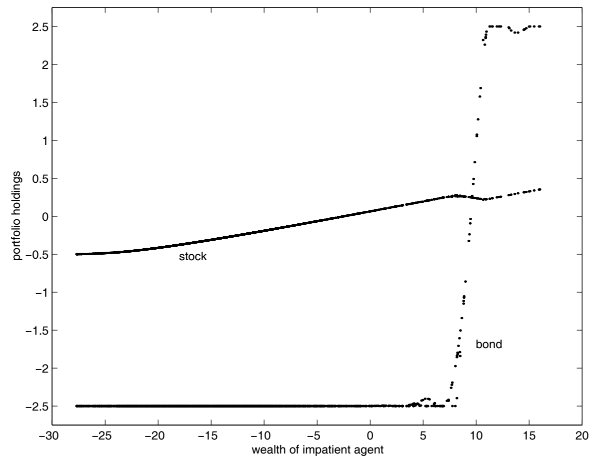


Figure 11. Portfolio holdings and wealth levels of the impatient agent

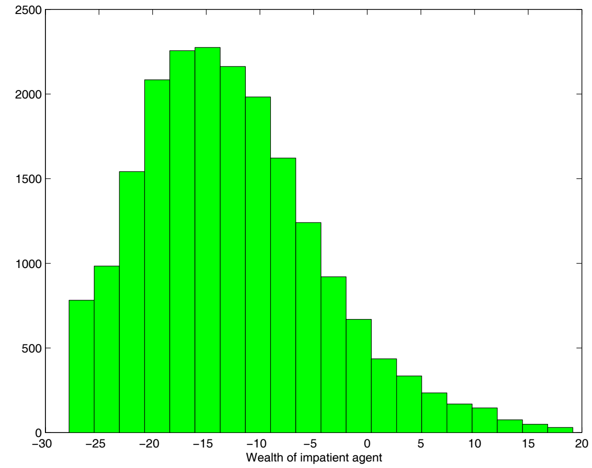
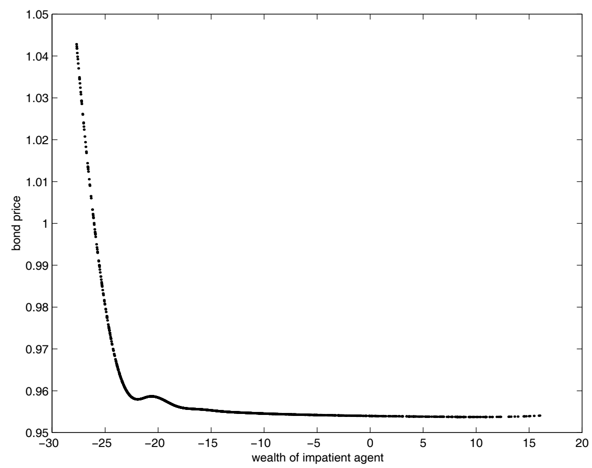


Figure 12. Histogram of wealth levels



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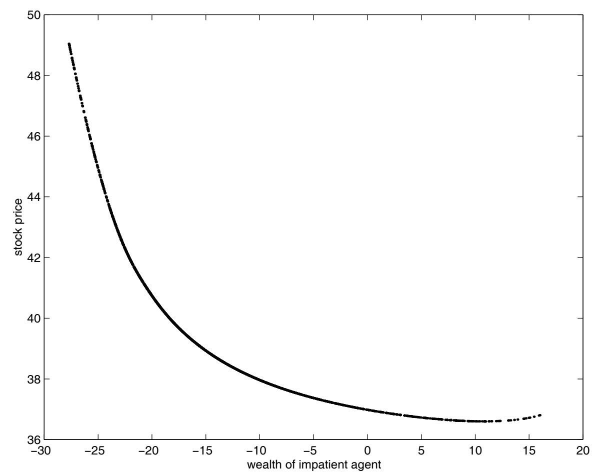

Figure 14. Stock price

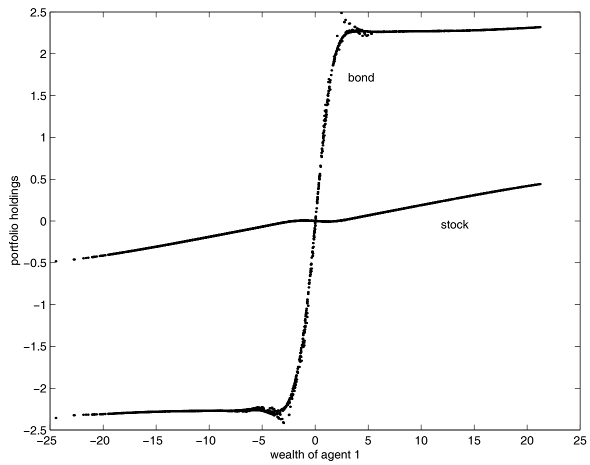

Figure 15. Portfolio holdings

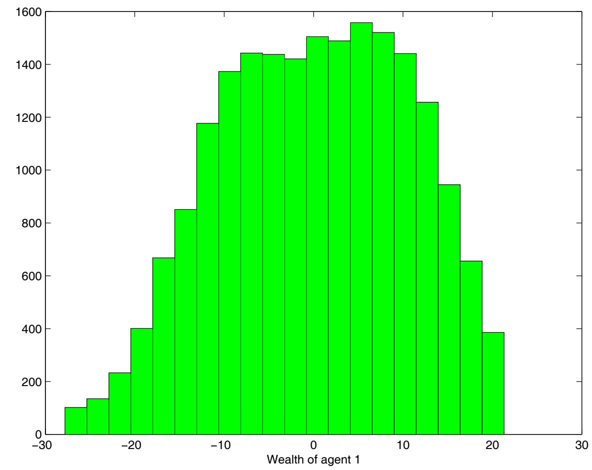

Figure 16. Histogram of wealth levels

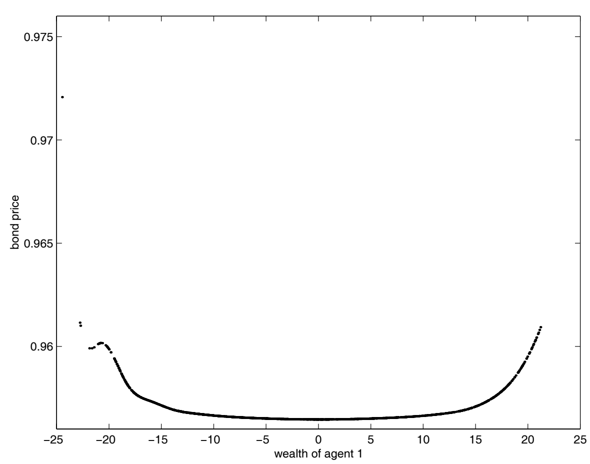

Figure 17. Bond price

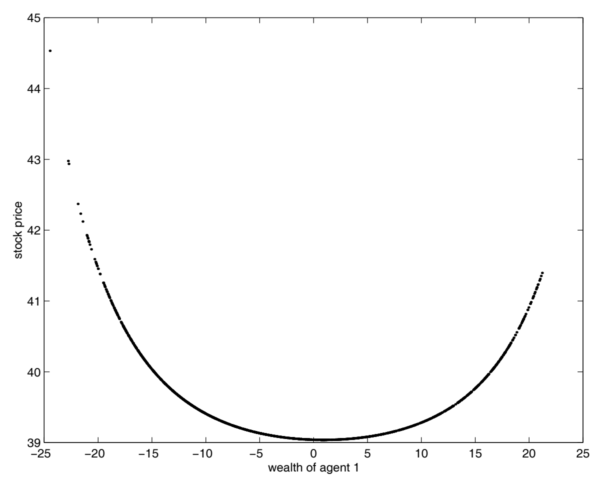

Figure 18. Stock prices

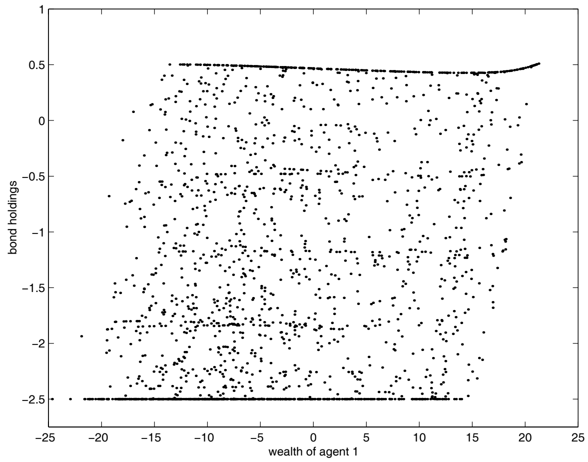

Figure 19. Bond holdings

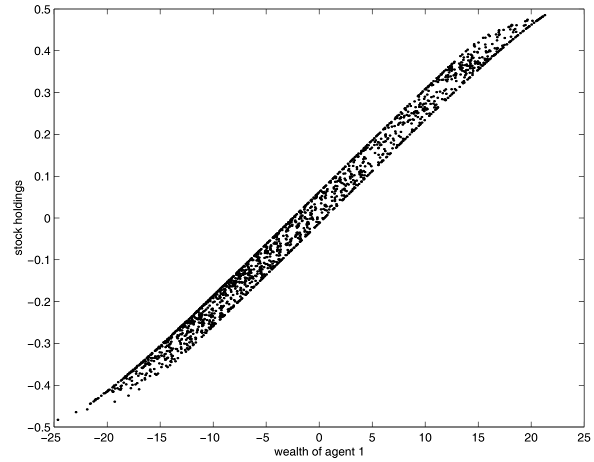

Figure 20. Stock holdings

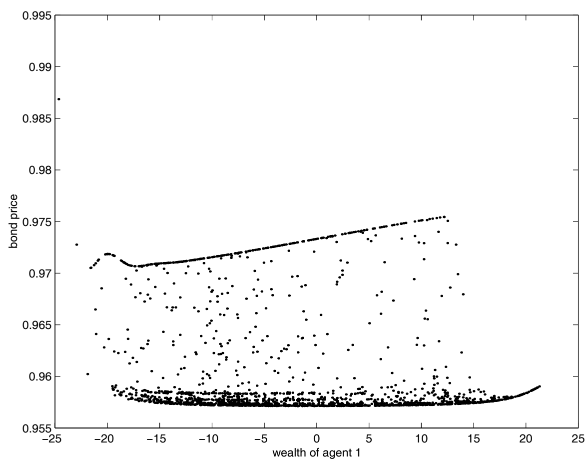

Figure 21. Bond price

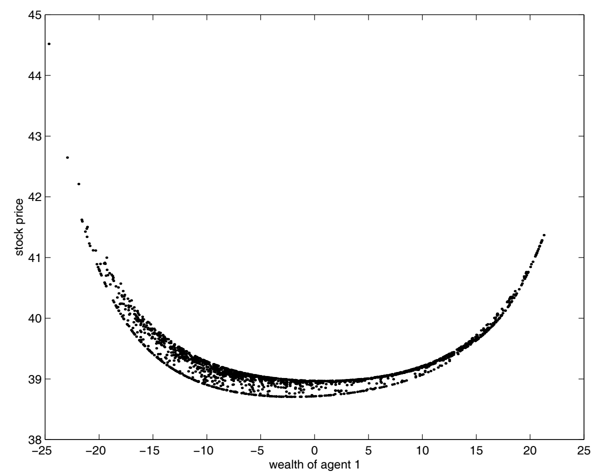

Figure 22. Stock prices