The Economic Effects of New Assets: An Asymptotic Approach

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ABSTRACT. General equilibrium analysis is difficult when asset markets are incomplete. We make the simplifying assumption that uncertainty is small and use bifurcation methods to compute Taylor series approximations for asset demand and asset market equilibrium. We apply these methods to analyzing the allocative and welfare effects of the introduction of derivative securities.

Precise analyses of equilibrium in asset markets is difficult since there are few cases that can be solved exactly for equilibrium prices and volume. Many analyses assume that markets are complete, implying that equilibrium is efficient and allowing artificial social planner's problems to replace equilibrium analysis. This approach is limited since it ignores transaction costs, taxes, and incompleteness in asset markets. This paper develops bifurcation methods to approximate asset market equilibrium without assuming complete asset markets. We begin from a trivial deterministic case where all assets have the same safe return and use local approximation methods to compute asset market equilibrium when assets have small risk.

The result is essentially a mean-variance-skewness-etc. theory of asset demand and equilibrium pricing. This approach is also more intuitive than the standard contingent state approach to equilibrium. The incomplete markets paradigm focuses on the difference between the number of states of nature and the number of assets. It is difficult to interpret such indices of incompleteness since it is difficult to count the number of contingent states in a real economy. Furthermore, one expects that the impact of asset incompleteness on economic performance is related more to the magnitude and statistical character of riskiness than to the number of contingent states. This paper's analysis focuses solely on the moments of the critical random variables, showing that they, not the number of states, govern the asymptotic properties of equilibrium. Since moments are more easily computed the result is a more practical approach to equilibrium analysis.

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Our approach is intuitive and similar in spirit to standard linearization and comparative static methods from mathematical economics. We begin with the no-risk case where we understand the equilibrium. We then use that information to compute equilibrium for nearby cases of risky economies. However, the resemblance is misleading. Usual linearization methods rely on the Implicit Function Theorem (IFT), which does not apply here because the critical Jacobian is singular. Instead, we must apply tools from singularity theory and bifurcation theory to solve our problem. Furthermore, we will need to compute higher-order approximations, not just the familiar first-order terms from linear approximation methods.

The technique can be used to analyze many issues of asset market equilibrium. The purpose of this paper is to present the key mathematical ideas and illustrate them with basic economic applications. We first apply bifurcation methods to derive approximations of asset demand. We then use these approximations of asset demand to compute approximations of asset market equilibrium. We compute asymptotically valid expressions for equilibrium for a variety of asset combinations, and compare how changes in asset availability affects equilibrium. We apply the results to solve an optimal spanning problem. In particular, we examine the welfare effect of adding a derivative asset, such as an option, to a market with only a stock and a bond, and compute that derivative which maximizes a social welfare function.

The idea of using expansion techniques to examine equilibrium with small risks is not new. Fleming (1971), and Fleming and Souganides (1986) explicitly take this approach. Magill (1977) computed a linear approximation for stochastic dynamic general equilibrium and showed how to use it to compute spectral properties of dynamic general equilibria. Much of the Real Business Cycle literature has relied on Kydland and Prescott's (1982) implementation of special cases Magill's method. Judd (1996) reviews several expansion methods and applications to economics.

However, most of these methods are direct applications of the Implicit Function Theorem (IFT). We show that the IFT cannot be directly used to approximate asset market equilibria. The problem is that all assets are perfect substitutes in the absence of uncertainty. This implies that there is a continuum of equilibrium asset allocations in the deterministic equilibrium and means that we do not know how to begin the expansion. Instead, we develop bifurcation methods to compute asset market equilibria when risks are small.

Samuelson(1970) examined asset demand with small noise, but in a different manner, and examined only expansions of asset demand. He essentially replaced the utility function with a low-order polynomial and noted that the solution to an asset demand problem with polynomial utility depended on the moments of the risky asset. Samuelson's approach generally requires solving nonlinear equations. He used these polynomial approximations to show that the linear quadratic approximation would not be as good as higher-order approximations but he argued that they would probably be adequate in actual economic problems. This paper gives examples where the linear-quadratic approximation would be unreliable, and higher-order approximations are necessary to answer critical questions.

In this paper, we identify the singularities which arise in small noise analysis and use bifurcation methods to handle them. We use these bifurcation methods to analyze both asset demand and asset market equilibrium. The bifurcation approach is particularly interesting since it handles the complete and incomplete asset market cases in the same way. This contrasts sharply with the conventional approach where the incomplete asset market case is far more complex than the complete market case (see Magill and Quinzii, 1996, for a more complete discussion). We can do this because we focus on the case of small risks. We compute asymptotically valid expansions by solving only linear equations of solving nonlinear equations as in Samuelson's method. We show that higher-order approximations are necessary to solve some problems. In particular, we compute the impact of a new asset on prices and portfolios, and show that it is determined by the skewness as well as means and variances of the assets. Finally, we compute the optimal derivative asset, showing that it is asymptotically equivalent to a quadratic option independent of tastes and returns.

Since our analysis makes no assumptions about the span of assets, it is also a method for computing equilibrium in some economies with incomplete asset markets. This is generally a difficult problem because the excess demand function is not continuous. Brown et al. (1996) and Schmedders (1998) have formulated algorithms for computing equilibria when asset markets are incomplete. Their methods aim to compute equilibrium for any such model. Our method is only valid locally but is much faster when it is valid since it relies on relatively simple and direct formulas.

We first present the theorems of local approximation theory. Taylor's theorem and the IFT are familiar theorems which form the basis of comparative static and comparative dynamic methods. The IFT requires that a critical Jacobian be nonsingular. We present bifurcation theorems which apply when the Jacobian is singular but higher-order derivatives are not degenerate. We then apply these theorems to various asset market problems where the risk is small.

1. Local Approximation Methods at Nonsingular Points

Local approximation methods are based on a few basic theorems. They begin with Taylor's theorem and the IFT for R^n, which we state in this section. We also review standard economics applications of the IFT. We will first state the basic theorems in this section, and then present the bifurcation theorems in the next section.

1.1. Taylor Series Approximation. The most basic local approximation is presented in Taylor's Theorem:

Theorem 1. *(Taylor's Theorem for R^n) Let $X \subseteq R^n$ and p an interior point of X. Suppose $f : X \to R$ is analytic in an open ball $B(p;r)$. Then for all $x \in B(p;r)$*

$$\begin{aligned} f(x) = f(p) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(p)\ (x_i - p_i) \\ + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(p)\ (x_i - p_i)\ (x_j - p_j) \\ \vdots \\ + \frac{1}{k!} \sum_{i_1=1}^{n} \cdots \sum_{i_k=1}^{n} \frac{\partial^k f}{\partial x_{i_1} \cdots \partial x_{i_k}}(p)\, (x_{i_1} - p_{i_1}) \cdots (x_{i_k} - p_{i_k}) \\ + \mathcal{O}\left(\|x - p\|\right)^{k+1} \end{aligned}$$

The Taylor series approximation of $f(x)$ based at p uses derivative information at p to construct a polynomial approximation. If f is analytic on some ball $B(p;r)$ then this approximation converges to f on $B(p;r)$ as n increases. The theory only guarantees that this approximation is good near p. While the accuracy of the approximation will decay as x moves away from p, this decay is often slow, implying that a finite Taylor series can be a good approximation for x in a large neighborhood of p. In our problems below, we will present diagnostics which will indicate how good the approximation is at any particular x.

1.2. The Meaning of "Approximation". We often use the phrase "$f(x)$ approximates $g(x)$ for x near p", but the meaning of this phrase is seldom made clear. One trivial sense of the term is that $f(x^0) = g(p)$. While this is certainly a necessary condition, it is generally too weak to be a useful concept. Approximation usually means at least that $f'(p) = g'(p)$ as well. In this case, we say that "f is a first-order (or linear) approximation to g at $x = p$". In general, "f is an n'th order approximation of g at $x = p$" if and only if

$$\lim_{x \to x_0} \frac{\| f(x) - g(x) \|}{\| x - p \|^n} = 0$$

1.3. Implicit Function Theorem and Applications. The next important tool is the IFT.

Theorem 2. *(Implicit Function Theorem) Let $H(x,y) : R^n \times R^m \to R^m$ be analytic and $H(x_0, y_0) = 0$. If $H_y(x_0, y_0)$ is nonsingular, then there is a unique analytic function $h : R^n \to R^m$ such that $H(x, h(x)) = 0$ for (x,y) near (x_0, y_0). Furthermore, the derivatives of h at (x_0, y_0) can be computed by implicit differentiation of the identity $H(x, h(x)) = 0$.*

The Implicit Function Theorem (IFT) states that h can be uniquely defined for x near zero by a relation of the form $H(x, h(x)) = 0$, whenever $H_y(x_0, y_0)$ is not singular. This allows us to implicitly compute the derivatives of h with respect to x as a functions of x. For example, the first derivative is

$$\frac{\partial h}{\partial x}(x_0) = -\left(H_y(x_0, y_0)\right)^{-1} H_x(x_0, y_0).$$

When we combine Taylor's theorem and the IFT, we have a way to compute a locally valid polynomial[1] approximation of a function $h(x)$ implicitly defined by $H(x, h(x)) = 0$.

The IFT has been used in many ways. Comparative static analysis is a familiar application for microeconomists. Linearizing a deterministic, discrete-time dynamic model around a steady state is also just an application of the IFT in ℓ^2. In that case, the invertibility condition in the IFT is equivalent to local uniqueness and existence.

1.4. Previous Small Noise Analyses. This approach is not new to the literature, but the approach we take differs in substance and formalism from previous efforts. One line of previous work is that taken by Fleming(1971) and Judd and Guu (1993). Fleming showed how to go from the solution of a deterministic control problem to one with small noise added to the law of motion, the remarkable finding being that the problem is formally a singular perturbation but can be solved by standard regular perturbation methods as long as the control law was unique in the deterministic problem; call this the *local uniqueness condition*. Specifically, he showed that if one was studying the problem

$$\begin{aligned} \max \quad & E\left\{\int_0^T e^{-\pi t}\pi(x,u)dt\right\} \\ & dx = f(u,x)dt + \epsilon\sigma(u,x)dz \end{aligned} \tag{1}$$

then to approximate the problem for ϵ small one could solve for the control law $u = U(x,t)$ of the $\epsilon = 0$ problem and then apply the IFT to Bellman's equation. A key detail was that the control law needed to be unique in the $\epsilon = 0$ case. Judd and Guu implemented this approach for infinite horizon problems, used Mathematica to approximate both the deterministic and stochastic problem, and showed that the Fleming procedure produces good approximations.

The problem discussed in Fleming and Judd-Guu was easy since it could be handled by the standard IFT. A less trivial problem was examined in Samuelson (1970).

[1]The derivative information could also be used to compute a Padé approximant, or other nonlinear approximation schemes. Judd and Guu (1993) and Judd (1998) examine both approaches. In this paper, we will stay with the conventional Taylor expansions.

He examined the problem of asset demand when riskiness was small. We will return to that problem below.

A third example of small noise analysis is Magill's (1977) analysis of what is now called real business cycles. Magill showed how to compute linear approximations to (1), use these approximations to compute spectra of the resulting linear model, and proposed that the spectra of these models be compared to empirical data on spectra. Kydland and Prescott(1982) focussed on the special case of Magill's method where the law of motion $f(u, x)$ is linear in (u, x) and $\sigma(u, x)$ is a constant, and partially implemented Magill's spectral comparison ideas by comparing variances and covariances of these linear approximations of deterministic models to the business cycle data. This special case of Magill's approach to stochastic dynamic general equilibrium has been important in the Real Business Cycle literature.

Gaspar and Judd (1997) show how to compute higher-order expansions around deterministic steady states. Similarly, we will compute high-order expansions. Unfortunately, the macroeconomics literature ignores higher–order terms in Taylor series expansions. In fact, Marcet(1995) states that "perturbation methods of order higher than one are considerably more complicated than the traditional linear-quadratic case ..." Our bifurcation analysis of asset market equilibrium shows that higher-order terms are essential to understanding some issues, and that they are easy to compute.

A fourth example which illustrates the importance of proceeding formally is Huffman(1987). He attempts to model the impact of a productivity shock on asset trading. Huffman begins with a deterministic OLG version of a Lucas-style asset pricing model, and used the IFT to compute how investor wealth is affected by a small change in the initial conditions of productivity and wealth. He then interpreted the impulse response functions in this deterministic model as asset trading volume generated by shocks in a stochastic model. While the IFT is appropriate in a context such as Fleming (1971), the local uniqueness condition is not satisfied in Huffman. Suppose that one has stocks and bonds. In the deterministic model which Huffman perturbs, all assets are perfect substitutes, the demand for such assets is indeterminate, and the IFT is not applicable. The reader of Huffman(1987) is left guessing about the asset structure Huffman is assuming for the stochastic model. It appears that he allows trade in only equity in the stochastic model, but then he compares the results to empirical facts which come from a world where bonds are present. Since the methods used in Huffman ignore the differences between stocks and bonds, they are incapable of accomplishing the purported objective of studying asset pricing and trade volume in a stochastic world with multiple assets.

More recently Tesar (1995) has used the KP-KPR approach to evaluate the utility impact on countries of opening up trade in assets. Some of her examples showed that moving to complete markets could result in a Pareto inferior allocation, a finding which contradicts the first welfare theorem of general equilibrium. Kim and Kim (1999) have shown that this approach will often produce incorrect results. These examples illustrate the need for using methods from the mathematical literature instead of relying on *ad hoc* approximation procedures.

This paper illustrates the critical mathematical structure of asset market problems with small risks, and develops the relevant mathematical tools. While the model analyzed below is simple, the basic approach is generally applicable.

2. Bifurcation Methods

Our asset market analysis will require us to approximate an implicitly defined function at a point where the conditions of the IFT do not hold, that is, when $H_y(x_0, y_0)$ is singular. In some cases, there is additional structure which can be exploited by bifurcation methods. This section presents simple versions of bifurcation theory.

2.1. Bifurcation in R^1.

Suppose that $H(x,\epsilon)$ is C^2. One way to view the equation $H(x,\epsilon) = 0$ is that for each ϵ it defines a collection of x which solves the equation. We say that ϵ_0 is a *bifurcation point* if the number of solutions x to $H(x,\epsilon) = 0$ changes as ϵ passes through ϵ_0. The simplest tractable case is summarized in the following theorem.

Theorem 3. *(Bifurcation Theorem for R) Suppose $H : R \times R \to R$, H is analytic, and $H(x, 0) = 0$ for all $x \in R$. Furthermore, suppose that for some $(x_0, 0)$*

$$H_x(x_0, 0) = 0 = H_\epsilon(x_0, 0),\ \ H_{x\epsilon}(x_0, 0) \neq 0.$$

Then there is an open neighborhood $\mathcal{N}$ of $(x_0, 0)$ and a function $h(\epsilon)$, $h(\epsilon) \neq 0$ for $\epsilon \neq 0$, such that h is analytic

$$H(h(\epsilon), \epsilon) = 0 \text{ for } (h(\epsilon), \epsilon) \in \mathcal{N}.$$

and $(x_0, 0)$ is a bifurcation point.

Proof. The strategy to prove this theorem follows the trick of "solving a singularity through division by ϵ" (see Zeidler, 1998, Chapter 8). Define

$$F(x,\epsilon) = \begin{cases} \frac{H(x,\epsilon)}{\epsilon}, & \epsilon \neq 0 \\ \frac{\partial H(x,0)}{\partial \epsilon}, & \epsilon = 0 \end{cases} . \tag{2}$$

Since $H(x, 0) = 0$ for all x, $H(x,\epsilon) = \epsilon F(x,\epsilon)$ and F is analytic in (x,ϵ). Since $H_x(x_0,0) = 0$,the IFT can't be employed for H. Since $0 = H_\epsilon(x_0,0)$, L'Hospital's rule implies $F(x_0,0) = 0$. Direct computation shows $H_{x\epsilon}(x,\epsilon) = F_x(x,\epsilon) + \epsilon F_{x\epsilon}(x,\epsilon)$, which implies $F_x(x_0, 0) = H_{x\epsilon}(x_0, 0) \neq 0$. Since $F_x(x_0, 0) \neq 0$, we can apply the IFT to F at $(x_0, 0)$. Therefore, there is an open neighborhood $\mathcal{N}$ of $(x_0, 0)$ and an analytic function $h(\epsilon), h(\epsilon) \neq 0$ for $\epsilon \neq 0$, such that

$$F(h(\epsilon), \epsilon) = 0 \text{ for } (h(\epsilon), \epsilon) \in \mathcal{N}.$$

which in turn implies

$$H(h(\epsilon), \epsilon) = 0 \text{ for } (h(\epsilon), \epsilon) \in \mathcal{N}.$$

If $H_{\epsilon\epsilon}(x_0, 0) \neq 0$, direct computation shows that $F_\epsilon(x_0, 0) = \frac{1}{2} H_{\epsilon\epsilon}(x_0, 0) \neq 0$. In this case, we have $h(0) = x_0$ and

$$\frac{dh(0)}{d\epsilon} = -[F_x(x_{0,}0)]^{-1} F_\epsilon(x_{0,}0) = -\frac{1}{2}[H_{x\epsilon}(x_{0,}0)]^{-1} H_{\epsilon\epsilon}(x_{0,}0)$$

■

The condition $H_{x\epsilon}(x_{0,}0) \neq 0$ is the key that allows us to apply the IFT to prove the existence of a unique nondegenerate curve passing through $(x_0, 0)$ such that $F(h(\epsilon), \epsilon) = 0$,and compute $h'(0)$. We can also derive information for $h^{''}(0)$; direct computation shows that

$$\begin{aligned} 3H_{x\epsilon}(x_0, 0)h^{''}(0) &= -[3h^{'}(0)H_{xx\epsilon}(x_0, 0)h^{'}(0) \\ &\quad +3H_{x\epsilon\epsilon}(x_0, 0)h'(0) + H_{\epsilon\epsilon\epsilon}(x_0, 0)] \end{aligned}$$

which implies a unique value for $h''(0)$ as long as $H_{x\epsilon}(x_0, 0) \neq 0$. Again, the condition $H_{x\epsilon}(x_{0,}0) \neq 0$ is the key to solving $h''(0)$. The existence of all higher derivatives of h also relies solely on the solvability condition $H_{x\epsilon}(x_{0,}0) \neq 0$.

We assumed $H_{x\epsilon}(x_0, 0) \neq 0$. The division-by-zero trick can be applied to problems with higher-order degeneracies. If $H_{x\epsilon}(x_{0,}0) = 0$ then $F_x(x_{0,}0) = 0$, and we cannot apply the IFT to F in the proof. But if $F_\epsilon(x_{0,}0) = 0$ and $F_{x\epsilon}(x_{0,}0) \neq 0$ we can apply the bifurcation theorem to F. We would be implicitly constructing $H(x, \epsilon) = \epsilon^2 G(x, \epsilon)$.for some G, and dividing by ϵ^2 to prove the existence of a nontrivial $h(\epsilon)$ in $H(h(\epsilon), \epsilon) = 0$.

2.2. Bifurcation in R^n : The Zero Jacobian Case. The foregoing focussed on one-dimensional functions h. We can also apply these ideas for functions of $x \in R^n$. The same trick works for Theorem 4; therefore, its proof is omitted.

Theorem 4. *(Bifurcation Theorem for R^n) Suppose $H : R^n \times R \to R^n$ is analytic, and $H(x, 0) = 0$ for all $x \in R^n$. Furthermore, suppose that for some $(x_0, 0)$*

$$H_x(x_0, 0) = 0_{n\times n} \tag{3}$$

$$H_\epsilon(x_0, 0) = 0_n \tag{4}$$

$$\det(H_{x\epsilon}(x_0, 0)) \neq 0 \tag{5}$$

Then there is an open neighborhood $\mathcal{N}$ of $(x_0, 0)$ and a function $h(\epsilon) : R \to R^n,\ h(\epsilon) \neq 0$ for $\epsilon \neq 0$, such that

$$H(h(\epsilon), \epsilon) = 0 \text{ for } (h(\epsilon), \epsilon) \in \mathcal{N}$$

Furthermore, h is analytic and can be approximated by a Taylor series. In particular, the first-order derivatives equal

$$h'(0) = -\frac{1}{2} H_{x\epsilon}^{-1} H_{\epsilon\epsilon}.$$

Theorem 4 assumes $H_x(x_0, 0)$ is a zero matrix. There are generalizations which only assume that $H_x(x_0, 0)$ is singular. We do not present that theorem here since it is substantially more complex to present and not needed below. See Zeidler for a more complete treatment of bifurcation problems.

3. Portfolio Demand with Small Risks

The key assumption we exploit is that risks are small. This is motivated not by any claim that actual risks are small, but is a reasonable assumption for three reasons. First, this assumption allows us to solve the problem without making any parametric assumptions for either tastes or returns. We will derive critical formulas for allocations and welfare in a parameter-free fashion. The results will tell us which moments of asset returns are important and which properties of the utility function are important for the case of small risks. Even though the precise results depend on the assumption of small risks, the results are suggestive of general results, and could provide counterexamples to conjectures. There is no reason why the results of a local analysis are limited to the case of small risks; only the proof strategy uses the small risk assumption.

Second, we can check the quality of the approximations in an economically meaningful way. We don't really expect investors to perfectly solve their optimization problems; instead, we expect investors to choose portfolios which nearly satisfy their first-order conditions. We can directly compute the magnitude of the errors in the first-order conditions and judge whether they are acceptably small. Even if we were to take a more robust numerical approach we would not solve the first-order conditions exactly; instead, we would find some choice which nearly solved the first-order conditions. Any test we use to judge the quality of standard numerical methods can also be used to judge the quality of our bifurcation approximations. Numerical experiments indicate that our Taylor series approximations are quite good even when the risks are not trivial.

Third, the period of time in our analysis is not meant to model an entire life, but rather the period of time between trades. Modern markets operate at high speed, and the presence of many high-volume, low-transaction cost traders makes it reasonable to assume that only a small amount of risk is borne between trading periods. This can be made more precise in dynamic asset market models, but we leave that as a future exercise.

3.1. Demand with Two Assets. We begin by applying the bifurcation approximation methods to asset market demand. Suppose that an investor has wealth W to invest in two assets. The safe asset yields R per dollar invested and the risky asset yields Z per dollar invested. If the investors have θ shares of the risky asset, final wealth is $Y = (W - \theta)R + \theta Z$. We assume that he chooses θ to maximize $E\{u(Y)\}$ for some concave utility function $u(\cdot)$.

Economists have studied this problem by approximating u with a quadratic function and solve the resulting quadratic optimization problem. The bifurcation approach allows us to examine this procedure rigorously and extend it. We first create a continuum of portfolio problems by assuming

$$Z = R + \epsilon z + \epsilon^2 \pi \tag{6}$$

where z is a fixed random variable and π is intended to represent a risk premium. Let μ be the probability measure for z and let (a, b) be the (possibly infinite) support. At $\epsilon = 0$, Z is degenerate and equal to R. We assume $\pi > 0$, the natural case where a risky asset pays a premium. Note that (6) multiplies z by ϵ and π by ϵ^2. Since the variance of ϵz is $\epsilon^2 \sigma_z^2$, this models the intuition that risk premia are the same order as the variance. This may seem to prejudge the results. That will not be a problem since the analysis will validate the implicit assumptions contained in (6). Without loss of generality, we assume that $W = 1$ and $E\{z\} = 0$. If we assumed $\sigma_z^2 = 1$ then ϵ would be the standard deviation. We do not make this normalization in our presentation since it is more natural to see exactly where variance enters the analysis.

The investor solves

$$\max_{\theta} E\{u(R + \theta(\epsilon z + \epsilon^2 \pi))\}$$

and the first-order condition for θ is

$$\epsilon E\{u'(R + \theta(\epsilon z + \epsilon^2 \pi))\,(z + \epsilon \pi)\} = 0. \tag{7}$$

The first-order condition in (7) is the familiar requirement that the future marginal utility of consumption be orthogonal to the excess return of the risky asset. The choice of θ is a function of ϵ implicitly defined by

$$H(\theta(\epsilon), \epsilon) = 0 \tag{8}$$

where

$$H(\theta, \epsilon) \equiv \int_a^b u'(R + \theta(\epsilon z + \epsilon^2 \pi))\,(z + \epsilon \pi)\, d\mu.$$

We want to analyze the solutions of (8) for small ϵ. However, $0 = H(\theta, 0)$ for all θ, because at $\epsilon = 0$ the assets are perfect substitutes each with a safe return of R. We do not know $\theta(0)$ since any choice of θ satisfies the first-order condition (7) when $\epsilon = 0$. Furthermore, $0 = H(\theta, 0)$ for all θ implies $0 = H_\theta(\theta, 0)$ for all θ, violating the nonsingularity condition in the IFT. Therefore, we cannot use the IFT directly to compute $\theta(\epsilon)$.

The situation is displayed in Figure 1. As ϵ changes, the equilibrium demand for the risky asset, θ, follows a path like ABC or like $DEGF$. Since the asset demand problem is a concave optimization problem there is a unique path of solutions to the first-order conditions whenever $\epsilon \neq 0$. At $\epsilon = 0$, however, the entire horizontal axis is also a solution to the equity demand problem. The path ABC crosses the θ axis vertically and represents a *pitchfork bifurcation*, whereas the path $DEGF$ crosses the θ axis obliquely and represents a *transcritical bifurcation*. In general, $H_{\epsilon\epsilon}(x_0, 0) \neq 0$ in Theorem 1 implies that the bifurcation is a transcritical bifurcation, and $H_{\epsilon\epsilon}(x_0, 0) = 0 \neq H_{\epsilon\epsilon\epsilon}(x_0, 0)$ implies a pitchfork bifurcation. The objective is to first find the bifurcation point, B or E, where the branch of equity demand solutions crosses the trivial branch of solutions to the first-order conditions, and then compute a Taylor series which approximates $\theta(\epsilon)$ along the nontrivial branch.

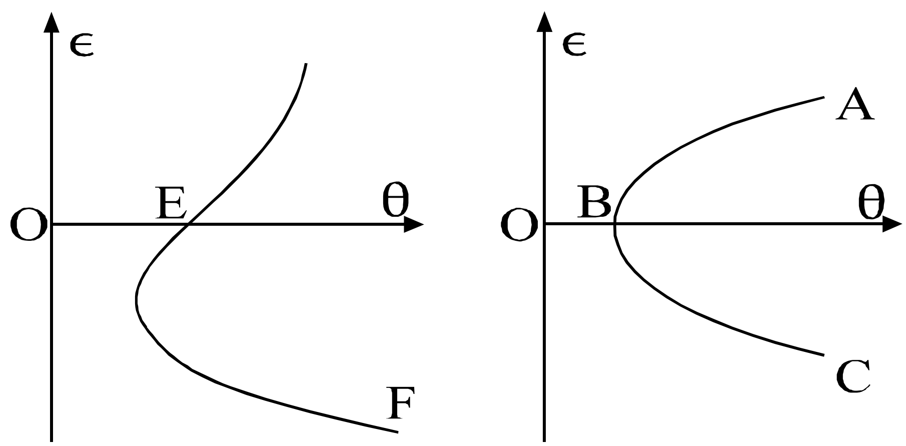

Figure 1: Bifurcation possibilities for asset demand problem

Computing θ_0. We proceed in an intuitive fashion to arrive at a solution validated by the Bifurcation Theorem. Since we want to solve for θ as a function of ϵ near 0, we first need to compute what θ value is the correct solution to the $\epsilon = 0$ case; specifically, we want to compute

$$\theta_0 \equiv \lim_{\epsilon \to 0} \theta(\epsilon)$$

Implicit differentiation of (8) with respect to ϵ implies

$$0 = H_\theta(\theta(\epsilon), \epsilon)\theta'(\epsilon) + H_\epsilon(\theta(\epsilon), \epsilon) \tag{9}$$

Differentiating (7) with respect to θ and ϵ we find

$$H_\epsilon(\theta, \epsilon) = E\{u''(Y)\,(\theta z + 2\theta\epsilon\pi)\,(z + \epsilon\pi) + u'(Y)\pi\}$$

$$H_\theta(\theta, \epsilon) = E\{u''(Y)\,(z + \epsilon\pi)^2\epsilon\}$$

At this point, we must check that the derivatives $H_\theta(\theta, \epsilon)$ and $H_\epsilon(\theta, \epsilon)$ exist. The problem is that we are claiming that

$$\begin{aligned} H_\epsilon(\theta, \epsilon) &= \frac{d}{d\epsilon}\left(\int u'(R + \theta(\epsilon z + \epsilon^2\pi))\,(z + \epsilon\pi)\; d\mu\right) \\ &= \int \frac{d}{d\epsilon}\left(u'(R + \theta(\epsilon z + \epsilon^2\pi))\,(z + \epsilon\pi)\right)\; d\mu \end{aligned} \tag{10}$$

is well-defined and true. The same is necessary for $H_\theta(\theta, \epsilon)$. We return to this issue after we compute a candidate bifurcation point.

We now apply the logic of the Bifurcation Theorem. At $\epsilon = 0$, $H_\theta(\theta, 0) = 0$ for all θ. The derivative $\theta'(0)$ can be well-defined in (9) only if $H_\epsilon(\theta,\, 0) = 0$ also. Therefore, we look for a bifurcation point, θ_0, defined by $0 = H_\epsilon(\theta_0,\, 0)$. At $\epsilon = 0$, this reduces to $0 = u''(R)\,\theta_0\sigma_z^2 + u'(R)\pi$, which implies

$$\theta_0 = -\,\frac{u'(R)}{u''(R)}\,\frac{\pi}{\sigma_z^2} \tag{11}$$

This is the simple portfolio rule indicating that θ is the product of risk tolerance and the risk premium per unit variance. If θ_0 is well-defined, then this must be its value.

Before we can apply the Bifurcation Theorem, we need to check that (10) holds at the solution in (11); in any specific case, we need to check this for the given utility function $u(c)$. We would like to pass the derivative operator through the integral. If μ is a discrete measure then (10) holds trivially. More generally, the following result from integration theory applies.

Lemma 5. : *Let $g(x, y) : R \times R \to R$ be differentiable in y, and define $G(y) = \int_a^b g(x, y)\; d\mu$. If $g_y(x, y)$ is bounded in an open neighborhood of $[a, b] \times y_0$, then $G(y)$ is differentiable at y_0 and*

$$\frac{dG}{dy}(y_0) = \int_a^b \frac{dg}{dy}(x, y_0)\; d\mu.$$

We make the following assumption to simplify our presentation

Assumption 1: *For all k and ℓ, the derivatives*

$$\frac{d^k}{d\epsilon^k}\frac{d^\ell}{d\theta^\ell}\left(u'(R+\theta_0(\epsilon z+\epsilon^2\pi))\,(z+\epsilon\pi)\right)$$

are uniformly bounded on some neighborhood of $\{(\theta,z,\epsilon)|\theta=\theta_0, z\in\mathrm{supp}(\mu),\epsilon=0\}$.

In our case, Lemma 6 applies to the analysis of (8) and (11).

Lemma 6. *If Assumption 1 holds then*

$$\frac{d^k}{d\epsilon^k}\frac{d^\ell}{d\theta^\ell}\left(H(\theta,\epsilon)\right)$$

exists and is computed by passing the differentiation operator through the integral sign.

Theorem 7 summarizes.

Theorem 7. *Suppose $u(c)$ is concave and analytic. Define θ_0 as in (11). Suppose Assumption 1 holds at θ_0. Then there is a function $\theta(\epsilon)$ which goes satisfies (8) such that $\theta(0)=\theta_0$.*

Proof. Direct application of the Bifurcation Theorem. ■

Computing $\theta'(0)$. Note that equation (11) is not an approximation to the portfolio choice at any particular variance. Instead, θ_0 is the limiting portfolio share as the variance vanishes. We will generally need to compute several terms in the Taylor series expansion for $\theta(\epsilon)$

$$\theta(\epsilon)=\theta_0+\theta^{'}(0)\epsilon+\theta^{''}(0)\frac{\epsilon^2}{2}+\theta^{'''}(0)\frac{\epsilon^3}{6}+.... \tag{12}$$

To get the linear approximation of $\theta(\epsilon)$ at $(\theta_0,0)$ we must go one more step to compute

$$\theta(\epsilon)\doteq\theta(0)+\epsilon\ \theta'(0). \tag{13}$$

To calculate $\theta^{'}(0)$ we need to do one more round of implicit differentiation. Differentiating (9) with respect to ϵ yields

$$0=H_{\theta\theta}\,\theta'\theta'+2H_{\theta\epsilon}\,\theta'+H_\theta\theta''+H_{\epsilon\epsilon}$$

At $(\theta_0,0)$,

$$H_{\epsilon\epsilon} = u^{'''}(R)\theta_0^2\, E\{z^3\}, \quad H_{\theta\theta} = 0, \quad H_{\theta\epsilon} = u^{''}(R)\, E\{z^2\}$$

Therefore,

$$\theta'(0) = -\frac{1}{2} H_{\theta\epsilon}^{-1} H_{\epsilon\epsilon} = -\frac{1}{2}\frac{u'''(R)}{u''(R)}\frac{E\{z^3\}}{E\{z^2\}}\; \theta_0^2. \tag{14}$$

Again, this uses Assumption 1 to establish the existence of the derivatives of H.

This formula tells us how the share of wealth invested in the risky asset changes as the riskiness increases, highlighting the importance of the third and second derivatives of utility and the ratio of skewness to variance. If the distribution of the risky asset is symmetric, then $E\{z^3\} = 0$, and the constant θ_0 is the linear approximation of $\theta(\epsilon)$. This is also true if $u'''(R) = 0$, such as in the quadratic utility case. The case of $\theta'(0) = 0$ corresponds to a bifurcation point like B in Figure 1. However, if the utility function is not quadratic and the risky return is not symmetrically distributed, then $\theta'(0) \neq 0$, and the linear approximation is a nontrivial function of utility curvature and higher moments of the distribution. This indicates that the bifurcation point is like E in Figure 1.

Dividing both sides of (14) by θ_0 implies

$$\begin{aligned} \frac{\theta'(0)}{\theta_0} &= -\frac{1}{2}\frac{u'''(R)}{u''(R)}\frac{E\{z^3\}}{E\{z^2\}}\;\theta_0 \\ &= \frac{1}{2}\frac{u^{'}(R)}{u''(R)}\frac{u'''(R)}{u''(R)}\;\frac{\pi}{\sigma_z^2}\;\frac{E\{z^3\}}{E\{z^2\}} \end{aligned} \tag{15}$$

Equation (15) expresses the relative change in equity demand as normalized skewness, $E\{z^3\}/E\{z^2\}$, and the risk premium increases.

Our formulas would be unintuitive and cumbersome if we expressed them in terms of $u(c)$ and its derivatives. Fortunately, there are some useful utility parameters we can use. Define the functions

$$\begin{aligned} \tau &\equiv -\frac{u'}{u''} \\ \rho &\equiv \frac{\tau^2}{2}\frac{u'''}{u'} = \frac{1}{2}\frac{u'}{u''}\frac{u'''}{u''} \\ \kappa &\equiv \frac{\tau^3}{3}\frac{u''''}{u'} = -\frac{1}{3}\frac{u'}{u''}\frac{u'}{u''}\frac{u''''}{u''} \end{aligned}$$

The function τ is the conventional risk tolerance. The bifurcation point is defined by

$$\theta_0 = \tau(R)\;\frac{\pi}{\sigma_z^2}$$

which expresses asset demand as the product of risk tolerance at the riskless consumption at the deterministic consumption, $\tau(R)$, and the price of risk, π/σ_z^2.

We will next derive a similar intuitive expression for $\theta'(0)$ using ρ. The utility terms in (15) indicates how preferences and asset returns interact to affect asset demand. The definition of $\rho(c)$ implies that (15) can be expressed as

$$\frac{\theta'(0)}{\theta_0} = \rho(R)\,\frac{\pi}{\sigma_z^2}\,\frac{E\{z^3\}}{\sigma_z^2} \tag{16}$$

This motivates our definition of *skew tolerance.*

Definition 8. *Skew tolerance at c is*

$$\rho(c) = \frac{1}{2}\frac{u'(c)}{u''(c)}\frac{u'''(c)}{u''(c)}$$

Skew tolerance has ambiguous sign since the sign of u''' is ambiguous, but standard intuition indicates that $u''' > 0$ is a natural presumption. If there is more upside potential than downside risk, then skewness is positive. If $u''' > 0$, an increase in skewness will cause asset demand to increase as riskiness increases. We suspect that investors prefer positively skewed returns, holding mean and variance constant. For example, $u''' > 0$ for the CRRA and CARA families of utility functions. We never assume this, but this presumption provides us with some intuition for the results.

There are many ways to manipulate the expression in (14), each leading to a different measure of how skewness affects asset demand. We chose the expression in (16) and the definition of skew tolerance because the result says that the relative change in θ as risk increases, $\frac{\theta'(0)}{\theta_0}$, equals the product of skew tolerance, the price of risk, and the magnitude of skewness normalized by variance. The term ρ appears in many important expressions below.

The linear approximation (14) will often not be sufficient for our purposes. To compute $\theta''(0)$ we begin by differentiating (9) with respect to ϵ at $\epsilon = 0$, yielding

$$3H_{\theta\epsilon}\theta''(0) = -(3H_{\theta\epsilon\epsilon}\theta'(0) + 3H_{\theta\theta\epsilon}(\theta'(0))^2 + H_{\epsilon\epsilon\epsilon}) \tag{17}$$

Equation (17) is linear in $\theta''(0)$. Since $H_{\theta\epsilon} \neq 0$, $\theta''(0)$ exists and equals

$$\theta''(0) = -\frac{3H_{\theta\epsilon\epsilon}\theta'(0) + 3H_{\theta\theta\epsilon}\theta'(0)^2 + H_{\epsilon\epsilon\epsilon}}{3H_{\theta\epsilon}}.$$

The second-order term $\theta''(0)$ can be written as

$$\frac{\theta''(0)}{\theta_0} = \left(\frac{(6\rho - 2)}{\sigma_z^2} \;+\; 4\rho^2\frac{E\{z^3\}^2}{(\sigma_z^2)^2}\right)\left(\frac{\pi}{\sigma_z^2}\right)^2 + \kappa\frac{E\{z^4\}}{\sigma_z^2}\left(\frac{\pi}{\sigma_z^2}\right)^2 \tag{18}$$

The expression in (18) depends on variance, skewness, and kurtosis statistics. However, the parameter κ interacts only with kurtosis. The term in (18) which involves kurtosis equals the product of κ, kurtosis normalized by variance, and the square of the price of risk. Therefore, κ expresses the sensitivity of asset demand to kurtosis. This motivates our definition of *kurtosis tolerance.*

Definition 9. *Kurtosis tolerance at c is*

$$\kappa(c) = -\frac{1}{3}\frac{u'(c)}{u''(c)}\frac{u'(c)}{u''(c)}\frac{u''''(c)}{u''(c)}$$

We could continue this indefinitely if u is locally analytic, an assumption satisfied by commonly used utility functions. Of course, the terms become increasingly complex. We end here since it illustrates the main ideas and these results are the only ones needed for the applications below. The general procedure is clear. Computing the higher-order terms is straightforward since they are all solutions to linear equations similar to (17). The components of (12) by a sequence of linear operations. This sequentially linear procedure is a substantial simplification relative to nonlinear alternatives, one of which is Samuelson's method.

Samuelson's Method. Samuelson (1970) also examined the problem of asset demand with small risks. His proposal was to replace $u(c)$ with a polynomial approximation based at the deterministic consumption; that is,

$$\begin{aligned} u(c) \;=\; & u(RW) \\ & +\epsilon R\theta E\{z\}u'(RW) \\ & +\epsilon^2\left(\frac{1}{2}R\pi\theta u'(RW)+\frac{1}{2}R^2\theta^2E\{z^2\}u''(RW)\right) \\ & +\epsilon^3\left(\frac{1}{2}R^2\pi E\{z\}\theta^2u''(RW)+\frac{1}{6}R^3\theta^3E\{z^3\}u'''(RW)\right) \\ & +... \end{aligned}$$

Since we assume $E\{z\}=0$, the first-order conditions for choosing θ when ϵ is small reduces to

$$0 = \frac{1}{2}R\pi u'(RW) + R^2\theta E\{z^2\}u''(RW)$$

which implies a choice of

$$\theta = -\frac{\pi u'(RW)}{2RE\{z^2\}u''(RW)}$$

However, the Samuelson method differs from ours for higher-order approximations. The next higher-order approximation would be computed by solving the third-order approximation of the first-order condition,

$$\begin{aligned} 0 &= \left(\frac{1}{2}R\pi u'(RW) + R^2\theta E\{z^2\}u''(RW)\right) \\ &+\epsilon\frac{1}{2}R^3\theta^2 E\{z^3\}u'''(RW) \\ &+... \end{aligned}$$

which is a quadratic equation. The bifurcation method computes an asymptotically valid approximation of the function $\theta(\epsilon)$, whereas the Samuelson approach eventually involves solving an approximate system at nonzero ϵ.

3.2. Demand with Three Assets. We have applied the R^1 version of the Bifurcation Theorem to the two-asset case. We next analyze the three-asset case to show the generality of the method and illustrate the key multivariate details.

Consider again our investor model but with three assets. The safe asset yields R per dollar invested and risky asset i yields Z_i per dollar invested, where $i = 1, 2$. Let θ_i denote the proportion of his wealth invested in risky asset i. The final wealth is $Y = (1 - \theta_1 - \theta_2)R + \theta_1 Z_1 + \theta_2 Z_2.$ Type i investors choose θ_i to maximize $E\{u(Y_i)\}$. To apply the Bifurcation Theorem, we assume that $Z_i = R + \epsilon z_i + \epsilon^2 \pi_i$. Without loss of generality, we assume that $E\{z_i\} = 0$, and define $\sigma_i^2 = E\{z_i^2\}$ be the variance of asset i's return and $\sigma_{12} = E\{z_1 z_2\}$ the covariance. We assume that the assets are not perfectly correlated; hence, $\sigma_i^2\sigma_j^2 \neq (\sigma_{ij})^2$.

The first-order conditions are

$$\begin{aligned} \epsilon E\left\{u'(Y)(\epsilon\pi_1 + z_1)\right\} &= 0 \\ \epsilon E\left\{u'(Y)(\epsilon\pi_2 + z_2)\right\} &= 0. \end{aligned}$$

The asset demand functions $\theta_i(\epsilon)$ are defined implicitly by

$$H(\theta_1, \theta_2, \epsilon) = \begin{bmatrix} H^1(\theta_1, \theta_2, \epsilon) \\ H^2(\theta_1, \theta_2, \epsilon) \end{bmatrix}$$

where

$$H^i(\theta_1, \theta_2, \epsilon) \equiv E\left\{u'(Y)(\epsilon\pi_i + z_i)\right\}.$$

Our goal is to solve θ_1 and θ_2 as functions of ϵ in some neighborhood of $\epsilon = 0$.

To invoke Theorem 4, we first compute

$$\begin{aligned} H_\theta(\theta_1, \theta_2, \epsilon) &= \epsilon u''(Y) \\ &\times \begin{bmatrix} E\{(\epsilon\pi_1 + z_1)^2\} & E\{(\epsilon\pi_1 + z_1)(\epsilon\pi_2 + z_2)\} \\ E\{(\epsilon\pi_1 + z_1)(\epsilon\pi_2 + z_2)\} & E\{(\epsilon\pi_2 + z_2)^2\} \end{bmatrix}. \end{aligned}$$

Obviously, $H_\theta(\theta_1, \theta_2, 0) = 0_{2\times 2}$. We proceed to computing a candidate bifurcation point for $H_\epsilon(\theta_1, \theta_2, 0) = 0$. Direct computation shows

$$\begin{aligned} H_\epsilon(\theta_1, \theta_2, 0) &= \begin{bmatrix} \pi_1 u'(R) + u''(R)\left(\theta_1\sigma_1^2 + \theta_2\sigma_{12}\right) \\ \pi_2 u'(R) + u''(R)\left(\theta_2\sigma_2^2 + \theta_1\sigma_{12}\right) \end{bmatrix} \\ &= u'(R)\begin{bmatrix} \pi_1 \\ \pi_2 \end{bmatrix} + u''(R)\Sigma\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}. \end{aligned}$$

where

$$\Sigma = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{bmatrix}$$

is the variance-covariance matrix of the risky components of the asset returns. The solution of the bifurcation equation $H_\epsilon(\theta_1, \theta_2, 0) = 0$ is

$$\begin{bmatrix} \theta_1(0) \\ \theta_2(0) \end{bmatrix} = -\frac{u'(R)}{u''(R)}\Sigma^{-1}\begin{bmatrix} \pi_1 \\ \pi_2 \end{bmatrix}$$

We need to verify the nonsingularity of $H_{\theta\epsilon}$ at $(\theta_1(0), \theta_2(0), 0)$. Direct computation shows that

$$H_{\theta\epsilon}(\theta_1(0), \theta_2(0), 0) = u''(R)\Sigma$$

for all θ_1, θ_2. The determinant of $H_{\theta\epsilon}$ at $(\theta_1(0), \theta_2(0), 0)$ is $u''(R)(\sigma_1^2\sigma_2^2 - (\sigma_{12})^2)$, which is nonzero as long as the assets are not perfectly correlated.

We have verified that all the conditions for Theorem 4 hold for our model. Hence, the bifurcation theorem ensures the existence of $\theta_1(\epsilon)$ and $\theta_2(\epsilon)$,at least in some neighborhood of $\epsilon = 0$. This procedure can be applied for an arbitrary number of assets.

4. Asset Market Equilibrium with One Risky Asset

We next apply bifurcation methods to analyze asset market equilibrium. We take the portfolio choice analysis of the previous section and turn it into an equilibrium analysis. We assume that two assets are traded. The safe asset yields R per dollar invested. Each share of the risky asset has a final value equal to $R(1 + \epsilon z)$, where z denotes a random variable with finite moments; we assume $E\{z\} = 0$ without loss of generality. For each value of ϵ we have an asset market with two assets; we call that economy the ϵ-economy. We assume two types of traders. Type i traders have initial endowments of a_i dollars and θ_i^e shares of the risky asset. The utility of a type i trader is $u_i(c_i)$ where c_i is the final wealth of type i traders. The price of the risky asset is fixed and the supply of the risky asset is fixed at the endowment $\theta_1^e + \theta_2^e$. Without loss of generality, we assume $\theta_1^e + \theta_2^e = 1$; this implies that z denotes aggregate risk in the aggregate endowment. We will compute the equilibrium price and allocation of the risky assets.

Let the price for the risky asset in the ϵ-economy be $p(\epsilon)$. Let θ_i and B_i be the shares of risky asset and safe asset owned by trader i after trading. The final wealth (and consumption) for trader i is

$$Y_i = \theta_i R(1+\epsilon z) + B_i R.$$

Trader i chooses θ_i to maximize his expected utility $E\{u_i(Y_i)\}$, where $u_i(.)$ is concave, subject to the budget constraint $\theta_i p + B_i = a_i + \theta_i^e p$. Market clearing implies $\theta_1 + \theta_2 = \theta_1^e + \theta_2^e$.

Type i investors' first-order condition for θ_i is

$$E\{u_i^{'}(Y_i)R(1+\epsilon z - p(\epsilon))\} = 0.$$

and his second-order condition is

$$E\{u_i''(Y_i)R^2(1+\epsilon z - p(\epsilon))^2\} < 0$$

always holds since u is concave. Market clearing implies $\theta_2 = 1 - \theta_1$. The investors' first-order conditions imply, for the ϵ-economy, the equilibrium conditions

$$H^i(\theta, p, \epsilon) = E\{u_i^{'}(Y_i)(1+\epsilon z - p(\epsilon))\},\ i = 1, 2$$

where $\theta = \theta_1$ and $\theta_2 = 1 - \theta_1$. The equilibrium problem is to solve θ and p as functions of ϵ in some neighborhood of $\epsilon = 0$. In other words, we want to compute $\theta(\epsilon)$ and $p(\epsilon)$ by exploring the equations

$$H^i(\theta(\epsilon), p(\epsilon), \epsilon) = E\{u_i^{'}(Y_i)(1+\epsilon z - p(\epsilon))\} = 0,\ i = 1, 2. \tag{19}$$

The IFT cannot be applied to analyze (19). If $\epsilon = 0$ then the safe and "risky" assets are perfect substitutes and must trade at the same price; hence, $p(0) = 1$. However, the value of $\theta(\epsilon)$ at $\epsilon = 0$ is indeterminate because

$$H^i(\theta, p, 0) = 0, \forall \theta.$$

We need to reformulate (19) so that we can apply Theorem 4. Implicit differentiation of $H^i(\theta(\epsilon), p(\epsilon), \epsilon)$ with respect to ϵ implies

$$H_\theta^i(\theta, p, \epsilon)\theta^{'}(\epsilon) + H_p^i(\theta, p, \epsilon)p^{'}(\epsilon) + H_\epsilon^i(\theta, p, \epsilon) = 0$$

$H_\theta^i(\theta, p(0), 0) = 0$ for all θ since $p(0) = 1$. Therefore, differentiability requires

$$\begin{aligned} 0 &= H_p^i(\theta, p, 0)p^{'}(0) + H_\epsilon^i(\theta, p, 0),\ i = 1, 2. \\ &= \left(E\{z\} - p^{'}(0)\right) u_i^{'}(c_i) = 0,\ i = 1, 2. \end{aligned}$$

where $c_i = (a_i + \theta_i^e)R$ is consumption in the no-risk case. Since $u_i^{'}(c_i)$ is never zero, this implies that $p^{'}(0) = E\{z\} = 0$ must hold if θ and p are differentiable at $\epsilon = 0$. Therefore, we have indeterminacy of $\theta(0)$ but there is only a single possible value for both $p(0)$ and $p^{'}(0)$. This prevents us from using Theorem 4 directly since the Jacobian matrix $H^i_{(\theta,p)}$ is not a zero matrix.

We want to solve for θ and p as a function of ϵ, yet, we need to impose $p(0) = 1$ and $p^{'}(0) = E\{z\} = 0$. Therefore, we express the risky asset's price as

$$p(\epsilon) = 1 + \epsilon E\{z\} + \frac{\epsilon^2}{2}\pi(\epsilon) = 1 + \frac{\epsilon^2}{2}\pi(\epsilon) \tag{20}$$

To apply Theorem 4, we solve the system

$$0 = H^i(\theta(\epsilon), 1 + \frac{\epsilon^2}{2}\pi(\epsilon), \epsilon) = \epsilon E\left\{u_i^{'}(Y_i)\left(z - \frac{\epsilon}{2}\pi(\epsilon)\right)\right\} \tag{21}$$

for $(\theta(\epsilon), p(\epsilon))$. We first divide $H^i(\theta, 1 + \epsilon E\{z\} + \frac{\epsilon^2}{2}\pi(\epsilon), \epsilon)$ by ϵ and focus on solving the system

$$\mathcal{H}^i(\theta, \pi, \epsilon) \equiv E\left\{u_i^{'}(Y_i)\left(z - E\{z\} - \frac{\epsilon}{2}\pi\right)\right\} = 0. \tag{22}$$

The functions $\mathcal{H}^i(\theta, \pi, \epsilon)$ now have the correct degeneracy since both $\mathcal{H}^i_\theta(\theta, \pi, 0) = \mathcal{H}^i_\pi(\theta, \pi, \epsilon) = 0$ for all (θ, π). Direct computation shows that the bifurcation point $(\theta(0),\ \pi(0))$ for (22) is defined by $\mathcal{H}^i(\theta, \pi, \epsilon) = 0$, $i = 1, 2$, and satisfies the linear equations:

$$\begin{aligned} -u_1^{'}(c_1)\pi(0) + 2Ru_1^{''}(c_1)\sigma^2\theta(0) &= 0 \\ u_2^{'}(c_2)\pi(0) + 2Ru_2^{''}(c_2)\sigma^2\theta(0) &= 2Ru_2^{''}(c_2)\sigma^2 \end{aligned} \tag{23}$$

where $c_i = (a_i + b_i)R$. The linear equations in (23) imply the unique candidate bifurcation point

$$\begin{aligned} \theta(0) &= \frac{\tau_1}{\tau_1 + \tau_2} \\ \pi(0) &= -2\frac{R\sigma^2}{\tau_1 + \tau_2} \end{aligned}$$

where τ_i is evaluated at $c_i = (a_i + b_i)R)$. We must check that the Jacobian matrix is nonsingular as required in Theorem 4. Here that Jacobian is

$$\mathcal{H}^1_{(\theta,\pi)\epsilon} = \begin{bmatrix} \mathcal{H}^1_{\theta\epsilon}(\theta(0), \pi(0), 0) & \mathcal{H}^1_{\pi\epsilon}(\theta(0), \pi(0), 0) \\ \mathcal{H}^2_{\theta\epsilon}(\theta(0), \pi(0), 0) & \mathcal{H}^2_{\pi\epsilon}(\theta(0), \pi(0), 0) \end{bmatrix} = \begin{bmatrix} Ru_1^{''}\sigma^2 & -\frac{1}{2}u_1^{'} \\ Ru_2^{''}\sigma^2 & \frac{1}{2}u_2^{'} \end{bmatrix}$$

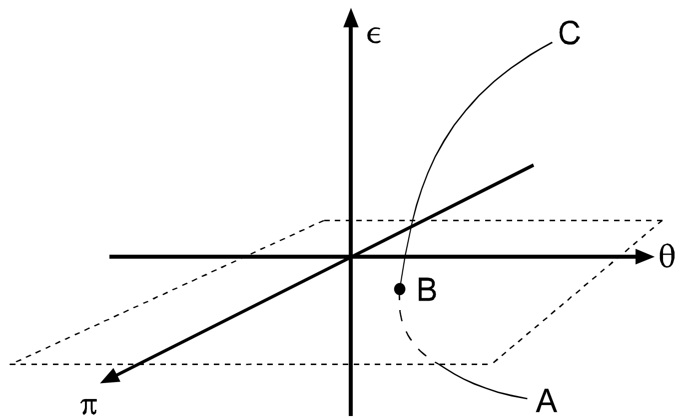

Figure 2: Bifurcation of Equilibrium Correspondance

Since the determinant of $\mathcal{H}^1_{(\theta,\pi)\epsilon}$ is $\frac{1}{2}R(u_1^{''}u_2^{'}+u_1^{'}u_2^{''})\sigma^2 \neq 0$, Theorem 4 can be applied to the representation of equilibrium in (21) and provides a proof of the existence and local uniqueness of $\theta(\epsilon), \pi(\epsilon)$.

This formulas for $\theta(0)$ and $\pi(0)$ are intuitive; the $u_i^{'}/u_i^{''}$ terms are the aggregate risk tolerances at $\epsilon = 0$, and the denominator is their sum, which is the social risk tolerance. The results are both very intuitive. The equilibrium risk premium is the ratio of the variance of risky wealth to the total risk aversion. Also, the fraction of equity held by a type 1 investor equals his contribution to social risk tolerance. These solutions resemble the results for the case of exponential utility and Normal returns.

Figure 2 displays the geometry of the bifurcation in (21). When $\epsilon = 0$, the entire $\theta - \pi$ plane constitutes an equilibrium. However, for nonzero ϵ we have a locally unique equilibrium. In Figure 2 the curve ABC represents the equilibrium manifold; the figure assumes uniqueness, a property which we will show in our problem but may not hold for similar problems.

Further implicit differentiations on $\mathcal{H}^i$ yield $(\theta'(0), \pi'(0))$ and other higher-order derivatives. They are

$$\theta'(0) = R\frac{\tau_1}{\tau_1+\tau_2}\frac{\tau_2}{\tau_1+\tau_2}(\rho_1-\rho_2)\frac{Cov(z,z^2)}{\sigma_z^2} \tag{24}$$

$$\pi'(0) = \frac{2R^2}{(\tau_1+\tau_2)^2}\left(\frac{\tau_1}{\tau_1+\tau_2}\rho_1 + \frac{\tau_2}{\tau_1+\tau_2}\rho_2\right)\frac{Cov(z,z^2)}{\sigma_z^2} \tag{25}$$

Equation (24) shows that type 1 investors increase their holdings of the risky assets depends on the skewness properties of the model. If their skew tolerance exceeds that of type 2 investors and skewness is positive, $Cov(z, z^2) > 0$, then they will increase their holdings as riskiness increases. Equation (25) shows that the risk premium will decrease (π is the negative of the risk premium) if there is positive skewness, and the magnitude of the increase depends on a weighted sum of the skew tolerances, where the weights are the $\epsilon = 0$ portfolio holdings, as well as the normalized skewness of the returns. Notice that we get these results for any utility function, not just for CRRA utility functions or other families which have $u''' > 0$.

Higher derivatives can be obtained by solving linear systems of equations. This approach is computationally simple to implement since it involves solving only a sequence of linear equations.

5. Asset Market Equilibrium with a New Asset

In the previous section, the market we discussed contained a safe asset and a risky asset. In this section, we will compare results for different asset spans. In particular, we introduce new assets into the market and compute asymptotically valid expressions for features of equilibrium. The results allow us to single out important "factors" for these expressions.

We assume that the new asset pays ϵRy and has price $q(\epsilon)$ in the ϵ-economy. We include the R factor so that y enters the problem in the same way as z. We force the payoff of the new asset to be zero when $\epsilon = 0$; hence, $q(0) = 0$. This implies no loss of generality since any portion of the asset's return which is deterministic conditional on ϵ will be equivalent to the bond, adding nothing to the asset span. We assume that the net supply of the new asset is zero since our main application will be to derivative securities. For instance, this new asset can be a call option with final value equal to $\max[0, \epsilon R(z - S)] = \epsilon R \max[0, (z - S)]$. Since the risky asset has value $R(1 + \epsilon z)$, this would be an option with strike price $R + \epsilon RS$. This may initially seem odd, but it is a standard option with strike price $R\,(1 + S)$ if $\epsilon = 1$. Again, the utility of each trader depends on his final wealth. We are interested in the equilibrium holdings and prices of the old risky asset and the new asset for each trader.

Let θ_i, B_i and ϕ_i be the shares of risky asset, safe asset and the new asset y owned by trader i after trading. The final wealth for trader i is

$$Y_i = \theta_i R(1 + \epsilon z) + B_i R + \phi_i \epsilon R y.$$

Trader i chooses θ_i and ϕ_i to maximize his expected utility $E\{u_i(Y_i)\}$, where $u_i(.)$ is concave. The budget constraint is $\theta_i p + B_i + \phi_i q = a_i + \theta_i^e p$.

The first-order conditions with respect to θ_i and ϕ_i are

$$\begin{aligned} E\{u_i'(Y_i)R(1 + \epsilon z - p(\epsilon))\} &= 0 \\ E\{u_i'(Y_i)R(\epsilon y - q(\epsilon))\} &= 0 \end{aligned} \qquad (26)$$

Equilibrium is defined by combining the first-order conditions with the market clearing conditions; we shall compute the equilibrium values for θ_i, ϕ_i, π,and q as functions of ϵ in some neighborhood of $\epsilon = 0$. Let θ and ϕ denote θ_1 and ϕ_1; hence $\theta_2 = 1-\theta$ and $\phi_2 = -\phi$. Similar to the analysis of previous section, $\theta(0)$ and $\phi(0)$ are indeterminate but $\pi(0) = 1$ and $q(0) = 0$.

Just as in the case of equilibrium with one asset, we need to determine an appropriate parameterization. We implicitly differentiate the four first-order conditions in (26) with respect to ϵ, finding that differentiability of q and p at $\epsilon = 0$ requires that $[E\{y\} - q'(0)]u_i'((a_i + \theta_i^e)R) = 0$ and $[E\{z\} - p'(0)]u_i'((a_i + \theta_i^e)R) = 0$. Therefore, if q and π are well-behaved, $q'(0) = E\{y\}$ and $p'(0) = E\{z\}$. We want to solve for θ, ϕ, p,and q as functions of ϵ, at least in some neighborhood of $\epsilon = 0$, yet, we need $p(0) = 1$, $p'(0) = E\{z\}$ and $q(0) = 0, q'(0) = E\{y\}$. We choose the following parameterization:

$$\begin{aligned} p(\epsilon) &= 1 + \epsilon E\{z\} + \frac{\epsilon^2}{2}\pi(\epsilon) \\ q(\epsilon) &= \epsilon E\{y\} + \frac{\epsilon^2}{2}\psi(\epsilon) \end{aligned}$$

With this parameterization[2] we can apply Theorem 4. The bifurcation point $(\phi(0), \theta(0), \pi(0), \psi(0))$ is computed by solving the following system of linear equations:

$$\begin{bmatrix} Ru_1''\sigma_y^2 & Ru_1''\sigma_{yz} & 0 & -\frac{1}{2}u_1' \\ Ru_1''\sigma_{yz} & Ru_1''\sigma_z^2 & -\frac{1}{2}u_1' & 0 \\ Ru_2''\sigma_{yz} & Ru_1''\sigma_z^2 & \frac{1}{2}u_2' & 0 \\ Ru_2''\sigma_y^2 & Ru_2''\sigma_{yz} & 0 & \frac{1}{2}u_2' \end{bmatrix} \begin{bmatrix} \phi(0) \\ \theta(0) \\ \pi(0) \\ \psi(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ Ru_2''\sigma_z^2 \\ Ru_2''\sigma_{yz} \end{bmatrix}.$$

The solution of this system of linear equations produces the bifurcation point and implies the limit portfolio allocations

$$\theta(0) = \frac{\tau_1}{\tau_1 + \tau_2}, \quad \phi(0) = 0$$

and the limit risk premia

$$\pi(0) = -\frac{2R}{\tau_1 + \tau_2}\sigma_z^2, \quad \psi(0) = -\frac{2R}{\tau_1 + \tau_2}\sigma_{yz}.$$

The existence of solutions $\phi(\epsilon), \theta(\epsilon), \pi(\epsilon), \psi(\epsilon)$ near the bifurcation point is established by Theorem 4. Furthermore, we may want to compute the first-order derivatives $\theta'(0)$, $\phi'((0)$, $\pi'(0)$, $\psi'(0)$, the second-order derivatives $\theta''(0)$, $\phi''((0)$, $\pi''(0)$,

[2] We are making a guess for a parameterization which allows us to apply our Bifurcation Theorem. More general versions of the Bifurcation Theorem would automatically construct the correct parametrization but we find it simpler to make a guess and then show that our Bifurcation Theorem can be applied to it.

$\psi^{''}(0)$), etc., which exist as long as the utility function is has sufficient derivatives. These higher derivatives can be obtained by solving linear systems of equations. Since the solutions are messy, we omit them except for the first-order derivatives.

The results follow standard intuition. For example, since $2R/(\tau_1 + \tau_2) > 0$, $\psi(0) > 0$ if and only if $\sigma_{yz} < 0$. The equilibrium price of the derivative asset is asymptotically equal to

$$q(\epsilon) = \epsilon E\left\{y\right\} - \epsilon^2 \frac{R}{\tau_1 + \tau_2} \sigma_{yz} + O(\epsilon^3), \tag{27}$$

which tells us that the derivative carries a positive risk premium (modelled here as a discount in the price) only if it is positively correlated with aggregate risk z.

5.1. Trading Patterns for the New Asset. We now determine the trading patterns for the new, zero-net-supply asset. Since $\phi(0) = 0$, we need to compute $\phi'(0)$ in order to determine the trading patterns for nonzero ϵ. The fact that $\phi(0) = 0$ does not mean that there is no trade, but just that it is of order less than ϵ. Direct computation shows that

$$\phi'(0) = R \frac{\tau_1 \tau_2 \left(\rho_1 - \rho_2\right)}{\left(\tau_1 + \tau_2\right)^3} \frac{\sigma_z^2 Cov(y, z^2) - \sigma_{yz} Cov(z, z^2)}{\left(\sigma_y^2 \sigma_z^2 - \sigma_{yz}^2\right)} \tag{28}$$

If $\phi'(0) > 0$, then trader 1 buys and trader 2 sells the new asset.. Since $\tau_i > 0$ and $\sigma_{yz}^2 < \sigma_y^2 \sigma_z^2$, the denominator is positive. Hence, the sign of $\phi^{'}(0)$ is determined by the remaining terms. This implies that the question of who buys and who sells the new asset depends on the traders' relative skewness tolerances, $\rho_1 - \rho_2$, and statistical properties of the assets.

Many of our results revolve around two indices, one based on tastes and one based on the statistical properties of the assets. We next define them.

Definition 10. *Let*

$$\Gamma = \rho_1 - \rho_2$$

denote the difference in skew tolerance between type 1 and type 2 investors. Let

$$\Psi = Cov(z, z)Cov(y, z^2) - Cov(y, z)Cov(z, z^2)$$

The next theorem relates the trading pattern in the new asset to our taste and statistical indices.

Theorem 11. *Suppose that* $\sigma_y^2 \sigma_z^2 - \sigma_{yz}^2 > 0$. *Then*

$$\phi'(0) > 0 \text{ iff } \Gamma\Psi > 0$$

We have decomposed the trading pattern into two terms. Γ is the difference in skew tolerances of the two investors. Ψ is more complex, but it is clearer in the case where y and z are uncorrelated. The case of $Cov(y, z) = 0$ is a natural one to consider since the only important part of y is that portion which is uncorrelated to existing assets including z. In that case, Ψ has the same sign as the covariance of y and z^2. Then, if the returns of y are higher when z deviates from its expectation (zero) and type i investors have a greater skew tolerance than type 2 investors, then type 1 investors buy the new asset y.

5.2. Change in stock holdings. The new asset causes the following change in agent one's holdings of the original asset. Let $\theta_z^b(\epsilon)$ and $\theta_z^a(\epsilon)$ denote the equilibrium holding of the old asset by type 1 investors before and after the introduction of the new asset. Then

$$\theta_z^a(\epsilon) - \theta_z^b(\epsilon) = R\frac{\tau_1\tau_2\left(\rho_1 - \rho_2\right)}{\left(\tau_1 + \tau_2\right)^3}\frac{\sigma_{yz}\left[\sigma_{yz}Cov(z, z^2) - \sigma_z^2 Cov(y, z^2)\right]}{\sigma_z^2\left(\sigma_y^2\sigma_z^2 - \sigma_{yz}^2\right)}\frac{\epsilon^2}{2} + O(\epsilon^3) \quad (29)$$

If y and z are uncorrelated, (29) reduces to zero, implying that the new asset has only $O(\epsilon^3)$ effects on the demand for the old asset. In general, the change in type 1 investors' holding of the old asset is inversely related to their holding in the new asset if $\sigma_{yz} > 0$ since (28) and (29) imply that $\theta_z^a(\epsilon) - \theta_z^b(\epsilon) = -\sigma_{yz}\phi'(0)$.

If the two assets are positively correlated, as would be the case with a call option, then the change depends on covariances and skewness. If σ_{yz} is small but positive, then type 1 investors decrease their holdings if their skew tolerance is larger and if $Cov(y, z^2) > 0$, that is, if the new asset is correlated with the tail of the old asset. In that case, the skew tolerant investors dump the old asset in favor of the new asset which has returns in the tail of the old asset. This is reversed only when σ_{yz} and $Cov(z, z^2)$ are large.

5.3. Price Effects of the New Asset. Our computations show that the equilibrium price for the risky asset remains unchanged up to the third order in its Taylor expansion. However, the fourth order term reveals the dominant effect of the new asset on the price of the old asset. We focus on the simple case of a new asset uncorrelated with existing assets.

Theorem 12. *Let $P_z^b(\epsilon)$ and $P_z^a(\epsilon)$ denote the equilibrium price of the first risky asset before and after the new asset being introduced. If $E\{yz\} = E\{y\} = 0$ then the price change is*

$$P_z^a(\epsilon) - P_z^b(\epsilon) = 4R^3\frac{\tau_1\tau_2\left(\rho_1 - \rho_2\right)^2}{\left(\tau_1 + \tau_2\right)^5}\frac{E\{yz^2\}^2}{E\{y^2\}}\epsilon^4 + O(\epsilon^5). > 0$$

In particular, the initial risky asset rises in value and rises more as the new asset is more correlated to the tails of the original asset.

More generally, the price change is

$$\begin{aligned} P_z^a(\epsilon) - P_z^b(\epsilon) &= 4R^3 \frac{\tau_1\tau_2\left(\rho_1 - \rho_2\right)^2}{\left(\tau_1+\tau_2\right)^5} \\ &\quad \times \left(\frac{\sigma_z^2 E\left\{yz^2\right\}^2 + 2\sigma_{yz}E\{z^3\}E\left\{yz^2\right\}}{\sigma_y^2\sigma_z^2 - \sigma_{yz}^2}\right)\epsilon^4 + O(\epsilon^5) \end{aligned}$$

Theorem 12 shows the elements which affect the impact on price of the original risky asset. The price change depends on third-order properties of the utility function. A simple linear-quadratic approximation approach is not adequate. Also, the price change depends on the covariance of the new asset with the extremes of the old asset. If it does not have some correlation with those extremes then there is no price change to the order ϵ^4. There may be a price effect but it would be an order of magnitude smaller.

Theorem 12 has a robust result: a nontrivial new asset will increase the price of the old asset. The magnitude of the price change depends on utility and returns but the sign of the dominant term is unambiguous. However, there is no price effect if the new asset is independent of the first asset. The price effect depends on the covariance between y and z^2, that is, the covariance between y and the extremes of z. The new asset complements the old asset in providing risk-sharing services. The new asset allows investors to allocate tail risk independent of the allocation of other risks. This makes the old asset more attractive.

5.4. Welfare Effects of the New Asset. We next derive the effect of a new asset on the welfare of each trader. Theory tells us that in one-good models such as ours, individual investors may gain or lose utility from adding an asset, but someone must gain. Our solutions will add some precision to those statements. The following theorem summarizes the result of our perturbation analysis.

Theorem 13. *Suppose there are two investors trading one safe asset returning R and one risky asset paying $R(1+\epsilon z)$. We add a new asset y which is uncorrelated to the existing assets. Assume further that $E\{z\} = 0$. Then the first-order welfare change from adding the asset y for trader 1, measured by the unit-free consumption equivalent $[U_1^a(\epsilon) - U_1^b(\epsilon)]/u_1'$, equals*

$$\frac{U_1^a(\epsilon) - U_1^b(\epsilon)}{u_1'} = \frac{R^4\tau_1^2\tau_2^2\left(\rho_1-\rho_2\right)^2}{2\left(\tau_1+\tau_2\right)^5}$$

$$\times\left(5\left(\frac{\theta_1^e}{\tau_1} - \frac{\theta_2^e}{\tau_2}\right) + \theta_2^e\right)\frac{E\left\{yz^2\right\}^2}{E\left\{y^2\right\}}\epsilon^4 + O(\epsilon^5)$$

The second trader's welfare change is symmetrically expressed.

With the derivatives computed by the bifurcation method, we can study the welfare effect of the new asset y. We shall study this issue by comparing each trader's utility functions before and after the availability of the new asset. Precisely, we shall expand the utility functions in terms of ϵ and examine the dominated term.

Let $U_i^b(\epsilon)$ and $U_i^a(\epsilon)$ denote trader i's optimal utility levels before and after the availability of y. The utility effect can be examined by $[U_i^a(\epsilon) - U_i^b(\epsilon)]/u_i^{'}$, which is a measure of the welfare change in terms of a consumption equivalent The total welfare effect on the market can be summarized by

$$\begin{aligned} \Delta SW &= \frac{U_1^a(\epsilon) - U_1^b(\epsilon)}{u_1^{'}} + \frac{U_2^a(\epsilon) - U_2^b(\epsilon)}{u_2^{'}} \\ &= \frac{R^4 \tau_1^2 \tau_2^2 (\rho_1 - \rho_2)^2}{2(\tau_1 + \tau_2)^5} \frac{E\{yz^2\}^2}{E\{y^2\}} \epsilon^4 + O(\epsilon^5) \end{aligned} \tag{30}$$

$$\begin{aligned} \frac{\Delta SW}{c} &= \frac{U_1^a(\epsilon) - U_1^b(\epsilon)}{c_1 u_1^{'}} \frac{c_1}{c} + \frac{U_2^a(\epsilon) - U_2^b(\epsilon)}{c_2 u_2^{'}} \frac{c_2}{c} \\ &= \frac{\tau_1^2 \tau_2^2 R^4}{(\tau_1 + \tau_2)^4} \frac{(\rho_1 - \rho_2)^2}{2(\tau_1 + \tau_2)} \frac{1}{c} \frac{E\{yz^2\}^2}{E\{y^2\}} \epsilon^4 + O(\epsilon^5) \end{aligned} \tag{31}$$

Since $\tau_i > 0$, the total welfare change is positive.

Theorem 14. *Social welfare as defined in ΔSW increases by adding any new asset in a market with small risks.*

Again, the result corresponds to simple intuition. The availability of a new asset makes the economy more efficient and increases total welfare. This result is different from other results in the literature but does not contradict them. Hart (1975) finds that adding an asset may reduce welfare. The model here has only one consumption good, and we examine only the case of a market with small risks.

5.5. The Three-investor Case. We next display the results for the three-investor case. We do this to indicate that the results are not special for the two-investor case. We also hope that these formulas indicate how the two-investor formulas generalize to the multiple-investor case in general. We will also examine only the case where all assets are orthogonal with unit variance, and where asset 2 is a derivative asset. We assume that the total endowment of asset 1 is unity. These normalizations simplify the formulas without losing any economic content.

The bifurcation point for investor one's equilibrium holding of asset j is

$$\theta_{1,j}(0) = \frac{\tau_1}{\tau_1 + \tau_2 + \tau_3}\Theta_j$$

where Θ_j is the endowment of asset j. The risk premium for asset j is

$$\pi_j(0) = \frac{2R}{\tau_1 + \tau_2 + \tau_3}\Theta_j$$

The first order impacts for asset allocations are

$$\begin{aligned}
\theta'_{1,1}(0) &= \frac{\tau_1\left(\tau_2(\rho_1 - \rho_2) + \tau_3(\rho_1 - \rho_3)\right)}{\left(\tau_1 + \tau_2 + \tau_3\right)^3} E\left\{z_1^3\right\} \\
\theta'_{1,2}(0) &= \frac{\tau_1\left(\tau_2(\rho_1 - \rho_2) + \tau_3(\rho_1 - \rho_3)\right)}{\left(\tau_1 + \tau_2 + \tau_3\right)^3} E\left\{z_1^2 z_2\right\}
\end{aligned}$$

These results say that as uncertainty increases, the equilibrium portfolios change in the direction of those who have the greatest skewness tolerance, where the importance of an investor type is proportional to his share of social risk tolerance.

The first-order price impacts are

$$\begin{aligned}
\pi'_1(0) &= 2\frac{\rho_1\tau_1 + \rho_2\tau_2 + \rho_3\tau_3}{\left(\tau_1 + \tau_2 + \tau_3\right)^3} E\left\{z_1^3\right\} \\
\pi'_2(0) &= 2\frac{\rho_1\tau_1 + \rho_2\tau_2 + \rho_3\tau_3}{\left(\tau_1 + \tau_2 + \tau_3\right)^3} E\left\{z_1^2 z_2\right\}
\end{aligned}$$

These results again say that the price changes are determined by skewness statistics and the social average skewness tolerance.

The utility effects are more complex. However, the critical fact is that the dominant effect on any investor's utility is proportional to the co-skewness statistic $E\left\{z_1^2 z_2\right\}$, and the impact of any social welfare function is also proportional to the same co-skewness statistic. These results show that the critical features of our two-investor analysis generalizes to the case of multiple investors.

6. The Optimal New Asset

We saw above that the dominant term in the total welfare change separates into a utility and endowment piece and a statistical term. This makes it possible to derive an "optimal" new asset which maximizes the total welfare change. We can apply the bifurcation theorem to solve the optimal asset problem.

For each new asset y and ϵ the equilibrium utility for investors of type i is $U_i(y, \epsilon)$. Any social welfare function takes the form

$$W(y, \epsilon) = \alpha U_1(y, \epsilon) + (1 - \alpha) U_2(y, \epsilon)$$

for some weight $\alpha > 0$. The optimal asset problem is

$$\max_y W(y, \epsilon) \tag{32}$$

Let $Y(\epsilon)$ be the optimal new asset for the ϵ economy. The function $Y(\epsilon)$ is defined by the solution to the first-order condition

$$0 = W_y(Y(\epsilon), \epsilon)$$

A Taylor series expansion for $Y(\epsilon)$ satisfies

$$0 = W_{yy}(Y(\epsilon), \epsilon)Y'(\epsilon) + W_{y\epsilon}(Y(\epsilon), \epsilon)$$

At $\epsilon = 0$ all new assets are redundant and have no effect on utility, implying that $0 = W_{yy}(Y(0), 0)$ for all y. We need to apply the bifurcation theorem in order to find $Y(0)$;in particular, $Y(0)$ solves the bifurcation equation

$$0 = W_{y\epsilon}(Y(0), 0) \tag{33}$$

Solving the bifurcation equation (33) is equivalent to solving

$$\max_y W_\epsilon(y, 0)$$

The solvability condition for the bifurcation equation (33) is

$$0 \neq \det(W_{yy\epsilon}(Y(\epsilon), \epsilon))$$

The second-order condition for the optimality problem (32) requires that $W_{yy\epsilon}(Y(\epsilon), \epsilon)$ be negative definite, implying solvability.

Equation (30) provides us with the asymptotic utility for any asset y. To find the asymptotically optimal asset, we want to find $y = f(z)$ which maximizes the dominant term in (30). More precisely, we want to find $y = f(z)$ which maximizes

$$\frac{\left(Cov(y, z)Cov(z, z^2) - Cov(z, z)Cov(y, z^2)\right)^2}{\left(\sigma_y^2\sigma_z^2 - \sigma_{yz}^2\right)\sigma_z^2}. \tag{34}$$

Since portfolios are linear combinations of assets, there are several normalizations we can make. First, we can scale y so that $\sigma_y^2 = 1$. Second, the only contribution which a new asset makes to welfare is that portion which is not correlated with preexisting assets; hence we can assume $E\{y\} = 0$ and $E\{yz\} = 0$. Then the welfare change (34) reduces to $Cov\left(y, z^2\right)^2$.

The optimal asset problem then reduces to the following (degenerate) calculus of variations problem.

$$\begin{array}{ll} \max_f & \left(\int f(z)z^2 d\mu(z)\right)^2 \\ s.t. & \int f(z)d\mu(z) = 0 \\ & \int zf(z)d\mu(z) = 0 \\ & \int f(z)^2 d\mu(z) = 1, \end{array} \tag{35}$$

where $\mu(z)$ is the distribution function of z. The solution to (35) must maximize the Lagrangian function

$$\begin{aligned} \mathcal{L} &= \left(\int f(z)z^2 d\mu\right)^2 + \lambda_1 \int f(z)d\mu + \\ & \lambda_2 \int zf(z)d\mu + \lambda_3 \left(\int f(z)^2 d\mu - 1\right) \end{aligned} \tag{36}$$

for some multipliers $\lambda_1, \lambda_2, \lambda_3$. We maximize (36) by solving for $f(z)$ at each z. This produces the first-order condition

$$2z^2 \left(\int f(z)z^2 d\mu\right) + \lambda_1 + \lambda_2 z + 2\lambda_3 f(z) = 0. \tag{37}$$

The first-order condition (37) implies that $f(z)$ must satisfy

$$f^*(z) = -\frac{2z^2 \left(\int f(z)z^2 d\mu\right) + \lambda_1 + \lambda_2 z}{2\lambda_3}$$

The multiplier λ_3 must be nonzero since a doubling of f would not violate any of the other constraint but would affect the objective. Therefore, f^* is a well-defined quadratic function[3] of the form $f(z) = \beta_0 + \beta_1 z + \beta_2 z^2$. Since the stock and bond together span the space spanned by the assets 1 and z, the key fact about the optimal new asset is that it introduces z^2 to the asset space. Therefore, any quadratic $f(z)$ will have the same impact on welfare. This proves our optimal asset theorem.

Theorem 15. *As riskiness goes to zero, any optimal new asset is asymptotically equivalent to adding $y = z^2$.*

7. CONCLUSION

We have used simple bifurcation approximation methods to examine simple asset market problems. The result is a mean-variance-skewness-etc. theory of asset market equilibrium. We can also apply these techniques to analyze the effects of introducing

[3] λ_3 must be nonzero since a doubling of f would not violate any of the other constraints but would affect the objective.

new assets. We have proven that the optimal new asset is asymptotically equivalent to introducing a quadratic option.

The analysis in this paper has focussed on simple problems with a single consumption good, two traders, and a few assets. The mathematical tools are general and can be applied to more complex problems. The results suggest that bifurcation methods can be used to analyze many problems.

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