# Solving Dynamic Stochastic Competitive General Equilibrium Models

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**Abstract**

The Scarf algorithm was the first practical, almost surely convergent method for computing general equilibria of competitive models. The current focus of much computational research is computing equilibrium of dynamic stochastic models. While many of these models are examples of Arrow-Debreu equilibria, Scarf's algorithm and subsequent homotopy methods cannot be applied directly since they have

an infinite number of commodities. Many methods have been proposed to solve dynamic models and some work well on simple examples. However, all have convergence problems and are not likely to perform as well in models with heterogeneous agents, multiple goods, joint production, and other features often present in general equilibrium models. This paper discusses weaknesses of standard methods for solving dynamic stochastic models. We then present an alternative Negishi-style approach that combines convergent methods for solving finite systems of equations with convergent dynamic programming methods to produce more reliable algorithms for dynamic analyses. The dynamic programming step presents the key challenge since most practical dynamic programming methods have convergence problems, but we argue that shape-preserving approximation methods offer a possible solution.

The Scarf (1967) algorithm was the first practical, surely convergent method for computing general equilibrium prices and, equivalently, systems of nonlinear equations in $\mathbb{R}^n$. This was followed by the development of almost surely convergent homotopy methods for solving nonlinear equations[1]. This work gave us reliable and efficient methods for solving finite-dimensional systems of equations. Economists are now interested in analyzing dynamic models and there is currently substantial effort on computing equilibrium of dynamic stochastic models. While many of these models are examples of Arrow-Debreu general equilibrium models, Scarf's algorithm and subsequent homotopy methods cannot be applied directly since dynamic stochastic mod-

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[1]See Eaves (1972), Garcia and Zangwill (1982), and Allgower and Georg (1990) for presentations of homotopy methods.

els involve an infinite number of commodities. Following the spirit of Scarf (1967), the focus of this paper is on current efforts to find efficient and convergent methods for solving dynamic economic models.

There is substantial activity on computing equilibria of dynamic stochastic models. See, for example, the papers included in the Taylor and Uhlig (1990) symposium, the description of projection methods in Judd (1992), and the surveys in Judd (1996, 1998) and Judd, Kubler, and Schmedders (2002). However, these methods do not meet the Scarf standard. Most current methods revolve around systems of Euler equations. These methods generally work well for simple cases, but sometimes fail to converge (or converge only after substantial tinkering) even for simple one-good, one-agent problems. Convergence is even less likely if applied to models with heterogeneous agents, multiple goods, multiple factors, and other features often present in general equilibrium problems.

We will proceed with the same goal displayed in Scarf (1967). Before Scarf, economists solved general equilibrium models with Newton's method and other available algorithms for solving systems of nonlinear equations. These often worked but were not globally convergent; for example, the basic convergence theorem for Newton's method assumes that one has a good initial guess. Dixon and Parmenter (1996) discusses these early methods in computable general equilibrium modelling. Of course, there were methods which would converge but they were impractical. Scarf (1967) points out that

Sperner's lemma suggests no procedure for the determination of an approximate fixed point other than an exhaustive search

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of all subsimplices until one is found with all vertices labeled differently. Clearly some substitute for an exhaustive search must be found if the problem is to be considered tractable..

The chief contribution of Scarf (1967) is the presentation of a tractable method to find the critical subsimplex. We face a similar problem today when we attempt to compute equilibrium of competitive dynamic models because the available methods either have convergence problems or are impractical.

This paper examines possible numerical strategies that combine convergent methods for solving finite systems of equations with convergent dynamic programming methods to produce algorithms that will be more reliable for solving competitive equilibria of dynamic stochastic models. We do not present a general convergence theorem but lay out the critical features necessary for efficient convergent methods. Specifically, we examine the Negishi approach for computing competitive equilibria of dynamic stochastic general equilibrium. The basic idea is simple: for each set of Negishi weights over a finite number of agents (or agent types) we solve a dynamic programming problem. The solutions to the dynamic programming problems imply price and consumption processes for each agent. Equilibrium requires that the value of the endowments equals the value of consumption plans. If we can solve the dynamic programming problem for arbitrary Negishi weights, then any conventional nonlinear equation method can be used to find a finite vector of Negishi weights where each agent is on his intertemporal budget constraint.

While this is a theoretically straightforward and standard idea (for exam-

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ple, Ginsburgh and Keyzer, 1997, discusses its application in some deterministic dynamic models), its implementation for stochastic dynamic problems presents numerical difficulties. In particular, there are few dynamic programming methods suitable for the special demands of this application. This paper discusses these numerical issues and the availability of numerically efficient and reliable algorithms for solving the critical dynamic programming step. We have the same goal that motivated Scarfs algorithm: find a reliable, robust, and relatively efficient algorithm for solving dynamic general equilibrium. Many of the surely convergent dynamic programming solution methods are too slow for this application since the Negishi method requires solutions to many dynamic programming problems. Furthermore, one needs accurate approximations not only of the value function but also of the gradients of the value function and the allocation policies they imply.

Unfortunately, standard solution methods for dynamic programming problems are either impractical or have convergence problems that make them unreliable. The key fact is that most dynamic programming methods either suffer from a curse of dimensionality or are unstable because of difficulties in preserving shape. Concavity of production and utility functions is a standard assumption in competitive equilibrium analysis. These concavity properties imply concavity of the value function of any social planner's dynamic programming problem. Most dynamic programming algorithms do not exploit this property of the value function and will often produce nonconcave value functions for concave problems. Failure of shape preservation can lead to instabilities in solving dynamic programming problems. We argue that any convergent algorithm will need to be aware of these shape preservation issues

and will need to use shape-preserving approximation methods in the dynamic programming step of the Negishi algorithm.

There is, as usual, a trade-off between speed and safety. The good news is that easy shape-preserving approximation methods are available for problems with one continuous dimension. These methods could also easily address problems with one continuous dimension (such as capital) and many discrete states (such as productivity levels) which naturally reduce to a finite collection of problems with one continuous dimension. We present an example that shows that the reliability of shape-preserving approximation comes at little cost in one dimension. However, the cost of shape-preserving strategies for higher-dimensional problems is nontrivial. There are some complex methods for two- and three- dimensional problems, and some costly methods available for preserving concavity in higher dimensions. While I not now aware of any practical and efficient method for multidimensional shape-preserving approximation, it is currently an active field of research in numerical analysis. As shape-preserving approximation methods are developed in the mathematical literature, economists can apply them to determine their practical value. In the meantime, economists will probably need to rely on Euler equation methods. Even though many problems can be solved by Euler equation methods, it is always valuable to develop reliable alternatives.

Section 1 presents a simple dynamic general equilibrium model, describes the conventional methods of solution, and discusses their weaknesses. Section 2 reviews the basic Negishi method for the static general equilibrium model. Section 3 presents the Negishi formulation for a general dynamic model. Section 4 discusses standard methods for solving dynamic program-

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ming problems and their weaknesses. Section 5 presents some simple shape-preserving approximation methods. Section 5.2 examines the performance of a shape-preserving method for a simple dynamic programming example.

# 1 A Dynamic General Equilibrium Model and Standard Solution Methods

To motivate the arguments below, we examine a simple dynamic stochastic model and the most popular methods for solving it. These methods are often thought to be very successful, but this perception is possibly due to the very simple models to which they have been applied and the extra, nonsystematic steps sometimes used to attain this success. There is little reason to believe that they will be successful in more general models.

To ease the notational burden we will examine the case of a single sector, multiple agent model. Let $u^i(c)$, $i = 1, ..., m$, be agent $i$'s utility function over consumption $c \in \mathbb{R}$ in each period, and assume a common constant discount factor $\beta$. Let $k_{i,t} \in \mathbb{R}$ be agent $i$'s ownership of capital at the beginning of period $t$ and let $K_t = \sum_{i=1}^{m} k_{i,t}$ denote the aggregate capital stock at the beginning of period $t$. We assume that output is CRTS in capital and labor. Let $F(K, \theta)$ be output (including the undepreciated capital stock) when total capital is $K$, labor input per capita is one, and productivity level is $\theta$. We assume that each agent supplies one unit of labor inelastically[2]; therefore, the marginal product of labor is $(F(K, \theta) - K F_K(K, \theta))/m$. Assume that

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[2]We assume inelastically supplied labor to reduce the notational burden; it is not essential to any of our arguments.

capital moves according to

$$K_{t+1} = F(K_t, \theta_t) - \sum_i c_{i,t}$$

$$\ln \theta_{t+1} = \rho \ln \theta_t + \varepsilon_{t+1}$$

where $\varepsilon_{t+1}$ is distributed i.i.d. over $t$. Since this model is recursive, equilibrium consumption of agent $i$ at time $t$ can be expressed as a function of the wealth distribution at time $t$, $k_t = (k_{1,t}, k_{2,t}, ..., k_{m,t})$ and current productivity, $\theta_t$; we let

$$c_{i,t} = C^i(k_t, \theta_t)$$

denote the equilibrium consumption policy functions. Most current methods for solving dynamic general equilibrium models focus on Euler equation representations of equilibrium. The Euler equations for this model are

$$u_i'(C^i(k, \theta)) = \beta E\left\{ u_i'(C_i(k^+, \theta)) r(K^+, \theta^+) | \theta \right\}, \ i = 1, 2, ..., m \quad (1)$$

$$k_i^+ \equiv Y_i(k, \theta) - C_i(k, \theta), \ i = 1, 2, ..., m$$

$$K \equiv \sum_i k_i, \ K^+ \equiv \sum_i k_i^+$$

$$\theta^+ = \theta^\rho e^\varepsilon$$

$$Y_i(k, \theta) \equiv k_i r(K, \theta) + w(K, \theta), \ i = 1, 2, ..., m$$

$$r(K, \theta) \equiv F_K(K, \theta)$$

$$w(K, \theta) \equiv (F(K, \theta) - K.F_K(K, \theta)) m^{-1}$$

where, if $(k, \theta)$ is today's state, $Y_i(k, \theta)$ is a type $i$ agent's income today, $w(K, \theta)$ is today's wage, $r(K, \theta)$ is today's rate of return on capital, $k_i^+$ is a type $i$ agent's wealth tomorrow, $K^+$ is tomorrow's aggregate capital, $\theta^+$ is tomorrow's productivity level, and $r(K^+, \theta^+)$ is tomorrow's rate of return on

capital. Equation (1) is a set of functional equations that must be satisfied in equilibrium. We will proceed, as does the dynamic general equilibrium literature, under the assumption that the stable solutions to (1) are locally unique.

Solution methods typically use parameterized families of functions to approximate $C^i(k, \theta)$. For example, linear approximations take the form

$$\widehat{C}^i(k, \theta; a) = \sum_{j=0}^{n} a_i \phi_i(k, \theta)$$

where the set $\{\phi_i | i = 1, 2, ..\}$ comprise a basis for the space of continuous functions over $(k, \theta)$. This formulation reduces the infinite dimensional problem to a finite-dimensional search for good choices of the $a$ coefficients. There are many different strategies available here. The tensor product method with orthogonal polynomials[3] approximates each consumption function as

$$\widehat{C}^i(k, \theta; a) = \sum_{j_1=0}^{n_k} \cdots \sum_{j_m=0}^{n_k} \sum_{\ell=0}^{n_\theta} a^i_{j_1...j_n\ell} \phi_{i_1}(k_1) \cdots \phi_{i_n}(k_m) \ \psi_\ell(\theta), \quad i = 1, ..., n$$

where the $\phi_i(k)$ are orthogonal polynomials over some interval $[k_m, k_M]$ and the $\psi_i(\theta)$ are orthogonal polynomials over the range of $\theta$. Other possibilities explored in the literature include complete orthogonal polynomials (Judd and Gaspar, 1997), multivariate splines (Judd et al, 1999, 2000), exponential polynomials (den Haan and Marcet, 1990), and Pade' functions (Judd and Guu, 1997). One could use neural networks, wavelets, or trigonometric polynomials, or one could construct problem specific bases (e.g., see the discussion of hybrid perturbation-projection methods in Judd, 1998). The

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[3]For more details, such as the possible notions of orthogonality, see Judd (1992) or Judd (1998).

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basic idea is to find some family of functional forms that produces a parsimonious approximation of $C(k, \theta)$ and is asymptotically complete in a space of functions which includes the equilibrium solution $C^i(k, \theta)$.

Once we have chosen some approximation scheme, we then need to fix the $a$ coefficients. There are many methods for determining the $a$ coefficients. We will briefly discuss them and their limitations.

Time iteration uses the Euler equation in an economically intuitive fashion to solve for $\widehat{C}(k, \theta; a)$. Time iteration picks a finite set $Z$ of $(k, \theta)$ points. Suppose the iteration $j$ approximation for type $i$ consumption is $\widehat{C}^i(k^+, \theta; a^j)$. In iteration $j+1$ we take each $(k, \theta) \in Z$, and solve the system of equations

$$u_i'(c_i) = \beta E \left\{ u_i'\left(\widehat{C}^i\left(k^+, \theta; a^j\right)\right) r\left(K^+, \theta^+\right) | \theta \right\}, \; i = 1, 2, ..., n \quad (2)$$

$$k_i^+ \equiv Y_i(k, \theta) - c_i, \; i = 1, 2, ..., n$$

$$K^+ \equiv \sum_i k_i^+$$

for $c_i$, $i = 1, ..., m$. For a fixed $(k, \theta)$ point, (2) is a nonlinear equation in the consumption vector $c = (c_1, .., c_m)$. Intuitively, it is similar to a static general equilibrium model where the random price of consumption tomorrow is the marginal utility tomorrow constructed by assuming that tomorrow agent $i$ will use the $\widehat{C}^i(k^+, \theta; a^j)$ consumption rule. Note that the choice of $c_i$ affects agent $i$'s wealth tomorrow but does not affect tomorrow's decision rule. Solving (2) for several choices of $z_\ell \in Z$ generates solutions $c^\ell$. These results are then used as data used to find $a^{j+1}$, either through interpolation or regression, so that the consumption functions $\widehat{C}^i(k, \theta; a^{j+1})$ approximate the $c^\ell$ solutions.

Successive approximation methods proceed more directly, using less com-

putation per step. Specifically, successive approximation also begins with some set of points $Z$, but now solves for the $c_i$ values in

$$
\begin{aligned}
u_i'(c_i) &= \beta E\left\{ u'\left(\widehat{C}^i\left(k^+, \theta; a^j\right)\right) r\left(K^+, \theta^+\right) | \theta \right\}, \ i = 1, 2, ..., m \quad (3) \\
k_i^+ &\equiv Y_i\left(k, \theta\right) - \widehat{C}^i\left(k, \theta; a^j\right), \ i = 1, 2, ..., m \\
K^+ &\equiv \sum_i k_i^+
\end{aligned}
$$

for a finite number of points $(k, \theta)$. The system (3) basically solves for consumption choices today taking as given tomorrow's marginal utilities if agents use the consumption rule $\widehat{C}^i\left(k^+, \theta; a^j\right)$ both today and tomorrow. This set of equations for $c_i^j$ is simpler to solve than the equations in time iteration since $c_i$ do not appear on the right-hand side. However, if were were to have multiple goods then there would be several Euler equations relating the current gradient of utility to future utility, creating a little general equilibrium problem for current allocation and production decisions. As with time iteration, the $c$ data is used to find the coefficients $a$ so that, for each $i$, $\widehat{C}^{i,j+1}(k, \theta; a)$ fit the $c_i$ data.

Successive approximation was used in the rational expectations model in Miranda and Helmburger (1988) who observed that it was an efficient method for computation. It was also motivated in den Haan and Marcet (1990) by learning arguments from Marcet and Sargent (1989). Successive approximation is often quite stable, converging to the equilibrium. For the case of a simple growth problem, Judd (1998, pages 557-8) shows that successive approximation is locally convergent except for some extreme choices of tastes and technology.

Projection methods (see Judd, 1992), such as Galerkin and collocation

methods, offer more a more general approach motivated by numerical considerations instead of economic tatonnement stories. They begin by first defining the residual functions

$$
\begin{aligned}
R^i(k, \theta, a) &= u'_i\left(\widehat{C}_i(k, \theta; a)\right) - \beta\, E\left\{u'\left(\widehat{C}_i\left(k^+, \theta^+; a\right)\right) r\left(K^+, \theta^+\right) | \theta\right\} \quad (4) \\
k_i^+ &\equiv Y_i(k, \theta) - C_i(k, \theta), \ i = 1, 2, ..., n \\
K^+ &\equiv \sum_i k_i^+
\end{aligned}
$$

A projection method then constructs a finite set of projections conditions

$$
P_{ij}(a) \equiv \int_{\theta_m}^{\theta_M} \int_{k_m}^{k_M} \cdots \int_{k_m}^{k_M} \widehat{R}^i(k, \theta; a)\, \psi_j(k, \theta)\, \omega(k, \theta)\, dk_1 \cdots dk_n d\theta
$$

where $i = 1, ..., n$, and $j = 1, ..., m$, $\omega(k, \theta)$ is a weighting function and the $\psi_j(k, \theta)$ are a set of test functions. Finally, a nonlinear equation solver is used to solve the projection equations for the coefficients $a$. This approach has the potential of being much faster than time iteration and successive approximation. For example, Newton's method converges quadratically. Newton-style methods may not be practical if $a$ is large but some combination of a block Gauss-Seidel method with Newton methods used within the blocks can bring some of the advantages of Newton's method to solving the large system. Time iteration and successive approximation methods are examples of projection methods since they use particular choices for projections and nonlinear equation solving methods.

These methods have proven successful for simple problems but will likely have problems for more general problems. There are three obvious problems. First, the issue of multiplicity raises several difficulties. Time iteration and successive approximations both revolve around solving a large number of

small artificial problems similar to static general equilibrium models. Solving these equations at any particular $(k, \theta)$ point in $Z$ is not a problem because one could use surely convergent methods, but multiple equilibria present coordination difficulties. Suppose that there were multiple equilibria to the dynamic model. Then there will possibly be multiple equilibria at some of the $(k, \theta)$ points we use. Since these problems are solved independently, there is no guarantee that the equilibrium selection will be consistent. This will not a problem with the Negishi method we present below since the constancy of the Negishi weights impose strong connections across choices in various $(k, \theta)$ states. In any case, the possibility of multiple equilibria in the dynamic economy means that we need to find all of the equilibria in the Euler equation problems and make consistent choices, a very difficult problem.

There is an even more fundamental problem presented by multiple equilibria. All of these methods assume the existence of a selection for equilibrium consumption $C(k, \theta)$ which is smooth in $(k, \theta)$. In more general models, we would be making the same assumption for price functions. This is not justified by general equilibrium theory. Each $(k, \theta)$ point corresponds to a different dynamic general equilibrium problem where $k$ is the initial endowment and $\theta$ is the initial productivity state. Regularity theory (see Debreu, 1976) tells us that the equilibrium manifold is smooth in endowments for *generic* endowments, but not at *all* endowments. The standard Euler equation formulation of the problem assumes that there is a smooth manifold which is an equilibrium selection map over *all* initial endowments. Since this is not true for some simple static general equilibrium problems, it is not a safe assumption for dynamic models.

Second, economists are often interested in the ergodic character of equilibrium. Unfortunately, ergodic properties will probably not be approximated well by standard methods. The approximations to consumption functions $C(k, \theta)$ will have errors that will accumulate over time. For example, we know that consumption and wealth will be perfectly correlated across individuals in the true equilibrium. Approximations to $C(k, \theta)$ will not have this property, and these errors will mean that the computed ergodic distributions for consumption and wealth may not be close to the true long-run distributions.

Third, we have no convergence theory for these methods. We would like to know that the sequence of $\widehat{C}_i(k, \theta; a^j)$ approximate solutions converge to the true $C_i(k, \theta)$ as we increase $j$ and the degree of the approximating polynomials (or splines or trigonometric polynomials, etc.) This is a difficult in infinite-dimensional nonlinear functional analysis.

Euler equation methods have been reliable for simple models but they have not been tested on more complex models. Since the focus of this paper is on reliability, we look elsewhere.

## 2 Reliable and Efficient Computation of General Equilibrium

We first review the basic ideas from the computational general equilibrium literature used in the next section's discussion of dynamic problems. Standard general equilibrium theory focusses on computing a zero of the excess demand function $E(p)$. The Scarf algorithm (see Scarf, 1967, 1973, and

1982) and homotopy methods can be applied directly to solving $E(p) = 0$.

If the number of goods and prices is large, the system $E(p) = 0$ is large. If the number of agents is much smaller than the number of goods, it is often desirable to use instead the Negishi method (also known as the planning method). The Negishi method exploits the first theorem of welfare economics, which states that any competitive equilibrium of an Arrow-Debreu model is Pareto efficient. Let $u^i(c^i)$ be agent $i$'s utility function over consumption $c^i \in \mathbb{R}^n$ and let $e^i \in \mathbb{R}^n$ be his endowment[4]. Therefore, for any equilibrium there is a set of nonnegative social welfare weights, $\lambda^i$, $i = 1, \ldots, m$, such that the equilibrium allocation of final consumption, $c^i, i = 1, \cdots, m$, is the solution to the social welfare problem

$$\max_{c^1, c^2, \ldots} \quad \sum_{i=1}^{m} \lambda^i u^i(c^i) \tag{5}$$

$$\text{s.t.} \quad \sum_{i=1}^{n} (e^i - c^i) = 0$$

The Negishi approach finds a set of social welfare weights, $\lambda^i$, $i = 1, \ldots, m$, such that the solution to (5) is an equilibrium allocation. Without loss of generality, we assume $\lambda \in \Delta \equiv \left\{ \lambda \geq 0 \,\middle|\, \sum_{i=1}^{m} \lambda^i = 1 \right\} \subset \mathbb{R}^m$.

The Negishi approach proceeds in a two-stage fashion. Given a vector of social welfare weights, $\lambda$, we compute the unique[5] allocation $(c^1, c^2, \ldots, c^m)$ which solves (5). As long as tastes are strictly concave and $C^2$, this is an easy optimization problem that can exploit the fastest optimization methods. Let $X(\lambda) = (X^1(\lambda), X^2(\lambda), \ldots, X^m(\lambda)) : \Delta \to \mathbb{R}^{m \times n}$ be the optimal allocation

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[4]We examine an endowment problem to keep the notation simple; all ideas apply to economies with production.

[5]For the purposes of this paper, we will assume strict concavity of all utility functions.

given Negishi weight vector $\lambda \in \Delta$. Since $u$ is concave, $X(\lambda)$ is continuous. The allocation $X(\lambda)$ implies a pattern of marginal rates of substitution that must equal the equilibrium prices if $X(\lambda)$ were an equilibrium allocation. These prices are defined by

$$p_j = \frac{u_j^1(X^1(\lambda))}{\sum_{\ell=1}^n u_\ell^1(X^1(\lambda))} \equiv P_j(\lambda). \tag{6}$$

In an equilibrium, each agent $i$ can afford his allocation, $X^i(\lambda)$, at the prices $P(\lambda)$. To check this we define the excess budget function

$$B_i(\lambda) \equiv P(\lambda) \cdot (e^i - X^i(\lambda)), \quad i = 1, \dots, m$$

If $B_i(\lambda)$ is nonnegative, then agent $i$ can afford $X^i(\lambda)$ at prices $P(\lambda)$ and have $B_i(\lambda)$ in unspent funds. The weights $\lambda$ correspond to an equilibrium if and only if $B_i(\lambda) = 0$ for each $i$. Therefore the Negishi approach reduces the equilibrium problem to solving the system of nonlinear equations

$$
\begin{aligned}
B_i(\lambda) &= 0, \quad i = 1, \dots, m-1 \\
\sum_{i=1}^m \lambda_i &= 1
\end{aligned}
\tag{7}
$$

for $\lambda \in \Delta$. Existence theory tells us that there exists a solution. Once we have a $\lambda \in \Delta$ that solves (7), we then take $P(\lambda)$ to be the equilibrium prices and $X(\lambda)$ the equilibrium consumption allocation.

The Negishi approach has substantial advantages if there are fewer agents than goods even though it adds new variables, $\lambda$, to the problem. The reason is that the equilibrium (and numerically more difficult) part of the problem, (7), is a nonlinear equation with a $m$ unknowns independent of the number of goods. Of course, the consumption bundles $c^i$ are also computed each time we

evaluate $B(\lambda)$, but that is done by a collection of $m$ concave optimization problems over $n$ variables. If $m$ is substantially smaller than $n$, as would be the case in dynamic problems with a finite, but long, horizon, then the nonlinear equation problem is much smaller in the Negishi approach than in a more direct approach focussed on solving $E(p) = 0$. If solutions to (5) can be computed in closed form, then this approach reduces to $m - 1$ equations in $m - 1$ unknowns in a simplex. More typically we need to use numerical methods to compute solutions to (5). This is not a substantial problem as long as the optimization method for solving (5) produces accurate answers.

Therefore, the Negishi approach replaces a possibly large system of excess demand conditions for equilibrium prices with a possibly much smaller set of nonlinear equations combined with a collection of well-behaved concave optimization problems. This approach works well for finite-dimensional problems. It also works well in deterministic dynamic problems since there are good methods for solving deterministic, concave dynamic problems; see Keyzer and Ginsburgh (1997) for a discussion of this case. We now focus on the Negishi approach for dynamic, stochastic problems.

# 3   A Negishi Approach to Stochastic, Dynamic General Equilibrium

We next present the Negishi approach to a simple stochastic dynamic model. Let $u^i(c^i)$ be agent $i$'s utility function over $n_c$ consumption goods[6], $c^i \in$

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[6]This formulation can include leisure. Let one of the components of $c$ be labor supply, $l$. Then the marginal utility of $l$ is negative and the "consumption expenditure" on $l$ is

$\mathbb{R}^{n_c}$, in each period. We assume a common constant discount factor $\beta$. Let $\theta_t \in \mathbb{R}^{n_\theta}$ be the productivity state in period $t$. We assume that productivity follows a stochastic law of motion; for specificity, we assume $\ln \theta_{t+1} = \rho \ln \theta_t + \varepsilon_{t+1}$ where the productivity shocks $\varepsilon_t$ are i.i.d. Let $K_t$ denote the vector of $n_k$ aggregate capital stocks at time $t$, and let $k_{i,0} \in \mathbb{R}^{n_k}$ be agent $i$'s initial endowment of capital stocks at the beginning of period $t = 0$. Assume that the convex production possibility set in period $t$ in productivity state $\theta_t$ is defined by $F\left(y_t, K_{t+1}, \widetilde{K}_t\right) \leq 0$ where $K_{t+1}$ is the capital available at the beginning of period $t + 1$ and $\widetilde{K}_t \leq K_t$ is the amount of capital actually used in production in period $t$. General equilibrium welfare theory again applies, telling us that for any equilibrium there is a set of nonnegative social welfare weights, $\lambda^i$, $i = 1, \ldots, n$, such that the equilibrium is, for some set of nonnegative weights $\lambda$, the solution to the social welfare problem

$$W\left(K_0\right) = \max_{c_t^i, \ y_t, \ \widetilde{k}_t} \quad \sum_{i=1}^{m} \lambda_i E\left\{\sum_{t=0}^{\infty} \beta^t \ u^i\left(c_t^i\right)\right\} \tag{8}$$
$$F\left(y_t, K_{t+1}, \widetilde{K}_t, \theta_t\right) \leq 0$$
$$\widetilde{K}_t \leq K_t$$
$$\sum_i c_t^i - y_t \leq 0$$
$$\ln \theta_{t+1} = \rho \ln \theta_t + \varepsilon_{t+1}$$

For any given set of nonnegative weights $\lambda$, the problem in (8) is a dynamic

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negative. This allows us to use our compact notation.

programming problem. The Bellman equation for (8) is

$$
\begin{aligned}
W\left(K,\theta\right) &= TW\left(K,\theta\right) \\
&\equiv \max_{c^i,\widetilde{K}} \sum_i \lambda_i\, u^i\left(c^i\right) + \beta E\left\{W\left(K^+,\theta^+\right)|\theta\right\} \\
& \quad F\left(y_t, K^+, \widetilde{K}, \theta\right) \le 0 \\
& \quad \widetilde{K} \le K \\
& \quad \sum_i c_t^i - y_t \le 0 \\
& \quad \ln\theta_{t+1} = \rho\ln\theta_t + \varepsilon_{t+1}
\end{aligned}
\tag{9}
$$

where $T$ is the Bellman operator. Since $T$ is a contraction operator (under mild assumptions on $u$ and $F$) there is a unique fixed point in the space of bounded functions to the Bellman equation $W = TW$. In particular, the sequence $W^j = TW^{j-1}$ converges to the solution $W$. In our discussion we will assume that we solve (9) via value function iteration; that is, the $W^j$ iterates are constructed by

$$
W^{j+1}\left(K,\theta\right) = TW^j\left(K,\theta\right)
\tag{10}
$$

Define

$$
Z\left(K,\theta\right) = \left(C\left(K,\theta\right), \widetilde{K}\left(K,\theta\right), Y\left(K,\theta\right), \mathcal{K}\left(K,\theta\right)\right)
$$

to be the vector of the equilibrium decision rules: the consumption allocation function $C^i\left(K,\theta\right)$ for each consumer, the capital utilization policy $\widetilde{K}\left(K,\theta\right)$, the output function $Y\left(K,\theta\right)$, and the gross saving function $\mathcal{K}\left(K,\theta\right)$ denoting

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the next period's aggregate capital stock. $Z(K, \theta)$ solves

$$Z(K, \theta) \equiv \arg \max_{c^i, \widetilde{K}, y, K^+} \sum_i \lambda_i \, u^i(c^i) + \beta E\{W(K^+, \theta^+) | \theta\} \quad (11)$$

$$F\left(y, K^+, \widetilde{K}, \theta\right) \leq 0$$

$$\widetilde{K} \leq K$$

$$\sum_i c^i - y \leq 0$$

$$\ln \theta^+ = \rho \ln \theta + \varepsilon_{t+1}$$

The solution to this problem implies a pattern of marginal rates of substitution that in turn implies a sequence of prices. Let good 1 be the numeraire; then the prices for goods at time $t$ is $p_t$, the vector of marginal rates of substitution with respect to the numeraire. Let $\Psi^i(K)$ be the current value at time $s$ of consumer $i$'s expenditure in periods in terms of commodity 1 at time $s$ if the economy has aggregate capital stock $K$ at time $s$. Since marginal utilities are proportional to prices, $\Psi^i(K)$ satisfies the recursive expression

$$\Psi^i(K, \theta) = u_1^i(c^i)^{-1}\left(u_c^i(c^i) \cdot c^i + \beta E\{u_1^i(c^{i,+}) \Psi^i(K^+, \theta^+)\} | \theta\right) \quad (12)$$

$$c^i \equiv C^i(K, \theta)$$

$$K^+ \equiv \mathcal{K}(K, \theta)$$

$$c^{i,+} \equiv C^i\left(\mathcal{K}(K, \theta), \theta^+\right)$$

where $u_c$ is the vector of marginal utilities with respect to the various consumption goods. Equation (12) is a linear integral equation in the unknown function $\Psi^i(K, \theta)$.

The lifetime budget constraint of agent $i$ includes the value of his initial wealth. Let $\psi_0(\lambda)$ be the price of capital relative to the numeraire at time

$t = 0$. For each agent $i$ define the excess budget function,

$$B^i(\lambda) = \Psi^i(K, \theta_0) - \psi_0(\lambda) k_{i,0} \tag{13}$$

Equilibrium is defined by the solution to

$$B^i(\lambda) = 0, \ i = 1, ..., m \tag{14}$$

$\lambda$ is any $\lambda$ such that $0 = B(\lambda) \equiv (B^i(\lambda))_{i=1}^m$. Equation (14) defines the Negishi approach to computing equilibrium.

We will assume that $B(\lambda)$ is well-behaved; that is, we assume that it has a finite number of zeroes and is smooth with respect to the parameters of the model. These regularity properties require some additional assumptions. The recent paper by Shannon (1999) gives a general statement on sufficient conditions for regularity of infinite-dimensional models of general equilibrium. Her result covers many interesting instances of our model.

**Determinacy Assumption:** $B(\lambda)$ *is Lipschitz continuous with finitely many zeroes.*

The Negishi approach has reduced equilibrium in an infinite-dimensional model to a finite set of equations defined on a finite number of welfare weights. With the determinacy assumption, we can safely proceed with computation if we can evaluate $B(\lambda)$. The key problem lies in efficiently computing $B(\lambda)$, a problem to which we now turn.

# 4 Dynamic Programming: Conventional Methods and Their Limitations

If the computer could do an infinite amount of mathematics, then the equations (11,12,13,14) express all that needs to be said. Unfortunately, the fixed point equation in (10) is an infinite dimensional problem since $W$ is a function of continuous variables $k$. The key decision in any dynamic programming algorithm is to find some way to approximate the $W^j$ functions. Once we have decided on some parametric family to use for $W^j$ approximations, we then apply (10) to these approximations. The expenditure function problem (12) is also an infinite-dimensional problem, but it is a linear Fredholm integral equation of the second kind for which there are many reliable solution methods. Similarly, once we have computed solutions to (10) and the expenditure function problem (12), the step for finding equilibrium $\lambda$ involves solving a finite system of nonlinear equations just as in the static case. The only difference between the static and dynamic cases is the extra effort in computing the efficient allocations and their cost conditional on $\lambda$. Therefore, we focus on solution methods for (10).[7]

There are several details that need to be decided to compute approximations to (10). Since we cannot deal directly with the space of continuous functions, we focus on a finite-dimensional subspace and approximate $W(K)$

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[7]There are other methods for solving dynamic programming problems. Policy iteration (a.k.a. Howard improvement) could also be used. Trick and Zin (1997) combine the linear programming approach to solving dynamic progrmming problems with spline approximations. While they may be better than value function iteration they both must face the same approximation issues.

with some parameterized set of functions. For example, we could choose some basis $\phi_i(K)$ for the space of continuous functions and some integer $N$, and use the approximation

$$W(K) \doteq \widehat{W}(K; a) \equiv \sum_{i=1}^{N} a_i \phi_i(K) \tag{15}$$

Numerical procedures then focus on the finite-dimensional task of finding a vector $a \in R^N$ such that $\widehat{W}(K; a)$ approximately solves (9). Examples of this approach are in Daniel (1976) and Johnson et al. (1993).

The basic task replaces $T$, a contraction mapping from continuous functions to continuous functions, with a finite-dimensional approximation, $\widehat{T}$, that maps functions of the form in (15) to functions of the same form. We construct $\widehat{T}$ in two steps. First, we choose a finite collection, $X$, of states $K$, and evaluate $w_i = (T\widehat{W})(K_i)$ at $K_i \in X$. We will refer to this as the *maximization step* since it is the maximization problem in (9) at $K_i$. The resulting $w_i$ values are data, called Lagrange data, we use to approximate the function $T\widehat{W}$. Since we want to stay in the family defined by (15), we use that data to find a value for $a$ such that $\widehat{W}(K; a)$ fits the $w_i$ data. This is the critical *approximation step*, and we denote the result $\widehat{T\widehat{W}}$. The nature of the approximation could be interpolation or it could be regression. In essence, $\widehat{T}$ takes a function $\widehat{W}$ of the form (15) and maps it to another function of the same form. We are now looking for a fixed point in the $N$-dimensional space of coefficients, or, equivalently, a fixed point of $\widehat{T}$ in the space of functions of form (15).

Dynamic programming also presents us with more information than is available in many approximation contexts. Standard value function approxi-

mation just focuses on finding the $w_i$ values in the maximization step. However, since the maximization step is an optimization problem with the parameter $K$, we can use the envelope theorem to easily compute the gradient of $(T\widehat{W})(K)$ at each $K_i$; denote this information as $\delta_i = \nabla(T\widehat{W})(K_i)$. Since this information is so cheap to generate, we would like to use it in any approximation scheme. The collection of $w_i$ and $\delta_i$ values constitute Hermite data.

Furthermore, we want to choose approximation schemes that help the algorithm proceed efficiently. For example, we want to choose a $\widehat{W}(K;a)$ approximation scheme which is smooth since that will help us solve the maximization step efficiently. Furthermore, we want the problem of finding the new coefficients $a$ to be easy.

We now face the critical challenge. $\widehat{T}$ is generally not a contraction mapping on the space of functions of the form (15). The construction of an algorithm to solve (9) revolves around choices that make the algorithm go fast versus choices that make $\widehat{T}$ a contraction map or nearly a contraction, map.

There are several ways to approximate the value functions in (9) but some are not suitable for use in this context. We need a method which not only solves for the value function in (9) but also accurately solves the consumption policy function, i.e., $c = C(k, \lambda)$ since they are used to compute prices and the excess budget functions $B^i(\lambda)$. This requirement makes it more difficult to find an approximation scheme that fits the needs of the method we use to find the zero of the excess budget system (7). For example, if we use a piecewise smooth homotopy method to solve (7) then we will need a

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method of (9) accurately approximate derivatives of the $B^i(\lambda)$ in (7), and accurate approximation of the derivatives of $B^i(\lambda)$ puts an extra burden on the method used to approximate $W$.

One basic approach for solving (9) is to discretize all state variables. This is ill-advised here for two reasons. First, the curse of dimensionality generally makes this approach impractical if there is more than one state variable. Second, even if one could compute a reasonable approximation of $W$ in (9) with a reasonable number of states, that does not mean that it can result in good approximation of the equilibrium $\lambda$. Under value function iteration, the sequence of value functions $W^j$ converge in the $L^\infty$ norm. This is comforting if we are interested in accurate computation of the value function. However, we care about the prices implied by the solution to (9), not the value function. The prices implied by any $W(k, \lambda)$ are related to the gradients of $W(k, \lambda)$, and the gradient error $\|\nabla W^i - \nabla W\|_\infty$ may be much larger than the value function error $\|W^i - W\|_\infty$. Errors in computing the value function's gradient will translate into errors in computing $B(\lambda)$ and even larger errors in solving $B(\lambda) = 0$. This problem may be particularly severe in cases where we discretize the state space where gradients $\nabla W$ can be no more than secants between two points in the discretization of $W$.

Approximation theory offers us many alternatives to discretization and are a key part of most numerical procedures. Polynomials could be used if the problem were smooth. Daniel (1977) and Johnson et al. (1993) argue for using splines in dynamic programming problems. There is a large variety of functional forms, including neural networks and rational functions, which could be used for dynamic programming problems.

Unfortunately, most standard function approximation methods have problems. The key problem is the lack of shape preservation. We know that the true value function $W(k, \lambda)$ is concave but many approximation methods will not produce concave approximants even when the data are consistent with concavity. Polynomial interpolation and spline interpolation methods can produce high quality approximations but they are particularly susceptible to failures of shape preservation. In fact, they can take concave monotone increasing data in one dimension and produce highly oscillatory interpolants. These failures of shape preservation means that $\widehat{T}$ is not generally a contraction map.

Matters only get worse in higher dimensions. These problems could possibly be overcome by taking sufficiently flexible approximations with sufficiently large amounts of data, but we do not know a priori how much flexibility will be needed to achieve a stable algorithm. That approach would be like saying that computing general equilibrium is no problem because if you take a sufficiently fine grid of the unit simplex and evaluate excess demand at the grid points, you will eventually find all the equilibria. This is the kind of exhaustive search that we want to avoid.

We should also mention the perturbation approach to solving (9). The first step is finding a steady state of the solution to (9) with zero variance. Hansen and Koopmans (1972) noted that the problem of finding an invariant capital stock of a deterministic optimal control problem, the term they used for what we now call the steady state of (9), can be solved by an application of Scarf's algorithm. The perturbation method then constructs Taylor series approximations of the solution to (9) around the deterministic steady state.

The first step is finding a linear approximation to the consumption policy function $C(k)$ around the steady state for the deterministic problem. Magill (1977) and Kydland and Prescott (1982) have described this procedure. One can achieve far more accurate approximations by computing a higher-order Taylor series approximation around the steady state, as is done in Judd and Guu (1993, 1997) and Jin and Judd (2002). These approximations include terms involving the variance of the disturbances. Taylor series expansions have many advantages. They are computed in a direct, noniterative fashion and need no initial guess or information other than the initial steady state. They are very good for states near the deterministic steady state. However, the accuracy of these approximations decays as one considers values for $k$ distant from the deterministic steady state and as one increase the volatility of exogenous shocks. Jin and Judd find that third- and fourth -order approximations are very good for capital stocks within 25% of the steady state, but even fifth- and sixth-order expansions are not good for $k$ equal to half of the steady state even if computing these expansions were feasible. Taylor series expansions constructed as in Jin and Judd (2002) will often be very competitive methods for solving (9), but their limited range of validity make them unreliable in general. In particular, if the aggregate capital stock at time zero differs substantially from the steady state aggregate capital stock then the perturbation approximation may not be valid along the transition path.

These points all indicate that it is difficult to solve the dynamic programming problem (9) in a reliable and efficient fashion using standard methods. The next section offers an alternative which will avoid many of the problems.

# 5 Shape-Preserving Approximation Methods

Since the value function is concave, shape preservation is a necessary component of any efficient and reliable computation method for dynamic economies. In this section we describe a shape-preserving approximation method that is available for problems with one continuous endogenous state variable and arbitrarily many discrete exogenous states. There are several methods for preserving shape in one-dimensional approximation of functions. In this section, we describe one simple approach due to Schumaker (1983). There are now many such techniques; Kvasov (2001) is a recent book which surveys the literature. We first examine the Hermite interpolation version and then discuss the Lagrange version.

## 5.1 Schumaker's Shape-Preserving Splines

Consider the shape-preserving Hermite interpolation problem on the interval $[x_1, x_2]$. We begin with the data $y_1, y_2, s_1, s_2$, and want to construct a piecewise-quadratic function $s \in C^1[x_1, x_2]$ that satisfies the four Hermite interpolation conditions.

$$s(x_i) = y_i, \ s'(x_i) = s_i, \ i = 1, 2. \tag{16}$$

We first examine a nongeneric case in which a single quadratic polynomial works. Specifically, if $(s_1 + s_2)/2 = (y_2 - y_1)/(x_2 - x_1)$, then the quadratic function

$$s(t) = y_1 + s_1(t - x_1) + \frac{(s_2 - s_1)(x - x_1)^2}{2(x_2 - x_1)} \tag{17}$$

satisfies (16). Straightforward computation shows that if the initial data are consistent with a concave shape then this quadratic polynomial is concave.

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In general, we won't be so lucky. The Schumaker method then adds a knot to the interval $[x_1, x_2]$ and constructs a spline with the desired properties. Schumaker provides formulas for a $\xi \in (x_1, x_2)$ such that there is a quadratic spline with nodes at $\xi$, $x_1$, and $x_2$ that satisfies (16). The general Hermite interpolation problem has data $\{(y_i, s_i, x_i) \mid i = 1, \cdots, n\}$. If the data is concave, then we can apply Schumaker's method to $[x_i, x_{i+1}]$ and preserve concavity. If we have Lagrange data, $\{(y_i, x_i) \mid i = 1, \cdots, n\}$, we must first add estimates of the slopes (Schumaker provides some simple estimates) and then proceed as we do with Hermite data. As long as the data is consistent with global concavity, we can produce a globally concave function. However, it is always better to use the true slopes if they are available.

## 5.2   Performance in a Simple Example

There is a legitimate concern that shape preservation comes at significant cost. Judd and Solnick (1994) presents evidence that shape preservation is practical in one-dimensional problems. They consider the optimal growth problem

$$\max \quad \sum_{t=0}^{\infty} \beta^t u(c_t)$$
$$k_{t+1} = f(k_t) - c_t$$

where $c_t$ is consumption in period $t$, $u(c)$ is the utility function at each date, $k_t$ is the capital stock at the beginning of period $t$, and $f(k)$ is the aggregate

production function in each period. They assume the specifications

$$
\begin{aligned}
u(c) &= \frac{c^{1-\gamma}}{1-\gamma} \\
f(k) &= k + \frac{(1-\beta)}{\alpha\beta}k^{\alpha}
\end{aligned}
$$

which imply that the steady state capital stock is $k = 1$.

Judd and Solnick (1994) solved this problem using several techniques: discretizing the state space, piecewise linear approximation for the value function, cubic spline approximation of the value function, polynomial interpolation of the value function, and the Schumaker shape-preserving quadratic spline approximation for the value function. They used the following parameter values: $\alpha = .25$, $\beta = .95$, $.99$, $\gamma = 10, 2, .5$ over the interval $k \in [.4, 1.6]$. They ran the discretization method using mesh sizes $\Delta k = .01, .001, .0001, .00001$. We take the solution to the $\Delta k = .00001$ discretization, which implies 120,001 discrete states over $[.4, 1.6]$, as the truth and compare other solutions to this one. They used both level and slope information when they applied the Schumaker shape-preserving method. The other methods use only level information. The polynomial interpolation method used Chebyshev zeroes (adapted for the interval $[.4, 1.6]$) since that is the optimal interpolation grid for polynomials.

Table 1 reports the relative errors in the consumption function for various methods. $N$ is the number of intervals used in spline methods and the degree of polynomial used in the polynomial method. Table 1 shows that linear interpolation is roughly an order of magnitude more accurate than the discrete method, and shape-preserving interpolation is at least another order of magnitude better. Cubic spline and polynomial interpolation meth-

ods were often better. However, they encountered the shape problems we hypothesized above. In fact, Table 1 shows that the fourth-order polynomial interpolation method did not converge (DNC). In general, if the range of $k$ is large or the curvature high then polynomial interpolation fails to preserve shape and value function iteration is unstable. Judd and Solnick also show that the time penalty of shape-preserving approximation is negligible.

These results make two important points. First, discretizing the state space is a very expensive way to proceed. Any of the interpolation methods achieved the same accuracy using much less computer time. This disadvantage surely grows with dimension. Second, the advantages of shape-preserving method come at small computational cost. The shape-preserving method was faster than linear interpolation methods which achieved the same accuracy, and they were often faster than polynomials and cubic splines with the same number of free parameters. The key question is how this generalizes to higher dimensions. Economists should follow the progress approximation theorists make in their continuing work on this problem.

## 5.3    Alternative Shape-Preserving Approximation Methods

The Schumaker method is just one of many one-dimensional methods. Unfortunately, there is no such rich set of choices for higher dimensions. This is an active area of research in approximation theory. There are some two-dimensional methods; Wang and Judd (2000) applied a two-dimensional approach to a portfolio problem. Also, one could discretize the exogenous variables, such as productivity, and then for each exogenous state use a different

low-dimensional shape-preserving spline for the endogenous states.

While we currently have no general efficient shape-preserving approxima-tion methods for higher dimension, there are some brute-force approaches for higher dimensions that immediately come to mind. For example, if $\Phi = \{\phi_1, \phi_2, ...\}$ were a set of concave functions, then so is any convex com-bination $\sum_i \alpha_i \phi_i$ where the $\alpha_i \geq 0$. Therefore, one could use regression methods where the $\phi_i$ are the regressors and the $\alpha_i$ are constrained to be nonnegative. In fact, there is no reason why one could not use overfitting procedures where the number of regressors exceed the number of data points. We are only trying to find some concave function that fits the data well. The shape-preserving approximation problem is regression in a cone, not a linear space.

Another possibility is to use the fact that any concave function equals the minimum of its tangent hyperplanes. That is, if $T(x, z)$ is the function linear in $x$ and tangent to $f(x)$ at $x = z$, then $f(x) = \min_z T(x, z)$. One possible approximation of $f$ is $\min_{z \in Z} T(x, z)$ where $Z$ is a finite set of points. This approximation would have kinks that would violate our objective of approxi-mating smooth functions with smooth functions. However, this problem can be avoided with exponential smoothing, as in

$$\widehat{f}(x; \alpha, \beta, \sigma) = -\sigma \ln \left( \sum_i e^{-\left(\alpha_i + \beta^i x\right)/\sigma} \right)$$

where the vectors $\beta^i$ and scalars $\alpha_i$ are free parameters chosen so that $\widehat{f}(x)$ is a good approximation of $f(x)$. Another possibility is to use concave functional

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forms familiar to economists, such as the CES functional form[8]

$$\widehat{f}(x; \alpha, \sigma, \gamma) = \left( \sum_i \alpha_i x_i^\sigma \right)^{\gamma/\sigma}$$

These are all cumbersome approaches to shape preservation and may prove useful in some cases. At least they demonstrate that shape preservation is not a hopeless goal. Shape-preserving approximation of higher dimensions is an active area of research which will hopefully produce far better methods.

# 6   Conclusions

We have outlined the some of the difficulties that will be encountered when one wants to construct reliable algorithms for solving dynamic general equilibrium models. Standard Euler equation methods would have many problems since they ignore issues of multiplicity of equilibrium. The Negishi method offers us a way to solve general dynamic models by reducing the problem to a series of dynamic programming problems. However, we need reliably convergent methods for the dynamic programming problems. Shape issues then become critical.

We have shown that shape-preserving dynamic programming methods are available at little computational cost for one dimensional problems. There are some shape-preserving methods available for two dimensions

The goal of this paper is to ask "Can we construct an algorithm for dynamic general equilibrium that surely converges to equilibrium?" While we have not accomplished this task to the extent Scarf (1967) succeeded, we

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[8]Professor Scarf made this suggestion in his discussion of this paper.

have identified the key difficulties. If one takes a Negishi approach then the key problem is finding a surely convergent method for the dynamic programming portion. Here shape preservation is the challenge. In low dimensional problems, that challenge can be met. Progress in approximation theory will hopefully allow us to tackle higher dimensions in the future.

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# Table

| Table 1: $L_2$ norm of relative errors in consumption | | | | | | |
|---|---|---|---|---|---|---|
| $N$ | | | $(\beta, \gamma)$ | | | |
| | (.95,-10.) | (.95,-2.) | (.95,-.5) | (.99,-10.) | (.99,-2.) | (.99,-.5) |
| Discretized model | | | | | | |
| 12 | 7.6e-02 | 2.8e-03 | 5.3e-03 | 7.9e-01 | 1.8e-01 | 1.1e-02 |
| 1200 | 1.0e-04 | 2.1e-05 | 5.4e-05 | 2.9e-03 | 5.4e-03 | 1.3e-04 |
| Linear Interpolation | | | | | | |
| 12 | 1.5e-03 | 9.8e-04 | 5.6e-04 | 1.5e-03 | 1.0e-03 | 6.3e-04 |
| 120 | 1.1e-04 | 3.7e-05 | 1.3e-05 | 1.4e-04 | 8.4e-05 | 4.2e-05 |
| Cubic Spline | | | | | | |
| 12 | 8.7e-05 | 1.5e-06 | 1.8e-07 | 1.3e-04 | 4.9e-06 | 1.1e-06 |
| 120 | 5.3e-09 | 5.6e-10 | 1.3e-10 | 4.2e-07 | 4.1e-09 | 1.5e-09 |
| Polynomial | | | | | | |
| 4 | DNC | 5.4e-04 | 1.6e-04 | 1.4e-02 | 5.6e-04 | 1.7e-04 |
| 12 | 3.0e-07 | 2.0e-09 | 4.3e-10 | 5.8e-07 | 4.5e-09 | 1.5e-09 |
| Shape Preserving Quadratic Hermite Interpolation | | | | | | |
| 4 | 4.7e-04 | 1.5e-04 | 6.0e-05 | 5.0e-04 | 1.7e-04 | 7.3e-05 |
| 12 | 3.8e-05 | 1.1e-05 | 3.7e-06 | 5.9e-05 | 1.7e-05 | 6.3e-06 |
| 40 | 3.2e-06 | 5.7e-07 | 9.3e-08 | 1.4e-05 | 2.6e-06 | 5.1e-07 |
| 120 | 2.2e-07 | 1.7e-08 | 3.1e-09 | 4.0e-06 | 4.6e-07 | 5.9e-08 |