Equilibrium Open Interest*

Kenneth L. Judd
Stanford University
Hoover Institution
Stanford, CA 94305
judd@Hoover.Stanford.EDU

Dietmar P.J. Leisen
McGill University, Faculty of Management
1001 Sherbrooke St. W.
Montreal, PQ H3A 1G5, Canada
Dietmar.Leisen@McGill.CA

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Abstract

Open interest in a financial contract describes the total number that are held long at the close of the exchange; it is a stylized fact that open interest across strikes peaks at-the-money. This paper sets up a two-period model to investigate agents' equilibrium holdings in a portfolio of call options and the resulting curve of open interest across strikes. We document that the shape of the open interest curve in complete markets is very sensitive to assumptions about preferences; in incomplete markets it is in addition to that also very sensitive to distributional assumptions and the strike grid. This is in sharp contrast to the observed robustness of the peak to changes in the underlying security and maturities.

Keywords

option demand, open interest, co-skewness, skewness preference

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1 Introduction

It is a stylized fact of option markets that call option open interest[1] peaks near to the at-the-money call option; that peak appears whatever the underlying stock or index and whatever the time-to-maturity. The literature focuses on the pricing of options under risk-sharing assumptions but largely ignores the demand in such contracts. This paper studies whether the robustness of the pattern is consistent with equilibrium trade in options.

We set up a series of two-period general equilibrium exchange economies and index them through a parameter ε that characterizes the amount of risk. Each economy consists of two agents that can trade a security and a portfolio of options on that security; thereby the absolute value of either agent's demand is open interest. We perform an ε-expansion of agent's equilibrium demand schedules. The leading terms determine the shape of the open interest curve when risk is small; we calculate them in closed-form to study the shape of the open interest curve under the assumptions of complete and incomplete markets, various distributions and various risk-preferences.

In the expansion the term of order zero corresponds to mean-variance tradeoffs; that term does not induce equilibrium demand in securities that are in zero net-supply[2]. The first order term corresponds to a generalized mean-variance-skewness analysis in which agent's preferences enter through skew-tolerance, a term that we define via a third derivative of agent's utility function. That term captures that agents typically care about asymmetric events like liquidity shocks and market crashes. Intuitively, such preferences lead to equilibrium trade in options in our economy, because call options are contingent claims that allow agents to alter the skewness of their portfolio payoff by trading events in the upper tail of the stock distribution.

Throughout we assume that the distributions of the underlying security have continuous

[1] Open interest in a financial securities denotes the total number of that contract that are held long at the daily close of the exchange; it is quoted at the end of each day for all financial contracts traded. We are interested in the shape across strikes with the same maturity. The peak appears for both calls and puts.

[2] This result is in line with mean-variance analyses: Under such preference structure two fund-separation rules hold; since options are in zero net-supply they are never part of the market portfolio and under such preferences options would therefore not be traded. Bates (2001) pointed out the importance of crash-aversion for option trade: he argues that the less crash-averse investors insure the more risk-averse investors. We make this explicit through skew-tolerance and third-order moments.

support; therefore markets are incomplete with any finite number of options. Preferences over skewness are the leading term in the expansion that induces equilibrium option demand. The relative size of that term is characterized by distributional characteristics and the strike grid, only. When only a small number of options can be traded we study that leading term, only, and document that the shape of the open interest curve is very sensitive to the type of distribution and the strike grid: there can be either a flat curve, or one with a peak or even a dip at the money. When the number of options is large such that the market is "almost complete" we find evidence that the leading term in the expansion of open interest is almost flat across strikes, independent of the distribution and agent's risk-preferences. In that case we therefore need to study higher order terms also to determine the shape of the open interest curve. We analyze these terms in complete markets prove that the leading terms that determine the shape of the open interest curve lead to a variety of shapes depending on risk-preferences.

The importance of contingent claims for risk-sharing is well known since Arrow (1953) and Debreu (1959). Ross (1976) pointed out that options complete the market; Green and Jarrow (1987), and Nachman (1988) discussed approximations of complete markets using contingent claims. Leland (1980), Brennan and Solanki (1981), and Franke, Stapleton and Subrahmanyam (1998) all studied who is buying or selling options but they do not provide quantitative results of option demand. Furthermore all these studies have relied on very specific assumptions, e.g. market completeness, lognormal shocks and that agents have HARA utility functions[3]. Our paper provides a toolbox within which the leading demand terms can be studied in closed form under changing assumptions. This allows us to study the shape of open interest across strikes in complete and incomplete markets for various distributions and risk-preferences.

Our analysis of open interest documents that both in complete and in incomplete markets the shape of the curve is very sensitive to changes in the distribution and risk-preferences. This contradicts the stylized fact that the peak is robust to changes in the underlying security

[3]Many finance papers focus on equilibrium pricing in a representative agent economy based on aggregation theorems in equilibrium setups (Rubinstein (1974), Constantinides (1982)). By its own nature the use of this tool eliminates equilibrium analysis of option demand. Given their importance the literature on option demand is sparse.

and time-to-maturity. The contribution of our paper is to point out the limits of the risk-sharing argument for option trading and pricing.

The remainder of the paper is organized as follows: Section 2 discusses the stylized fact about the shape of open interest across strikes. Section 3 describes our expansion, and section 4 derives the equilibrium allocation and relates co-skewness to demand (open interest). Section 5 discuss the open interest curves that result in incomplete markets both when the number of traded options is small and when it is large. Section 6 looks at open interest in the complete markets case and section 7 concludes the paper.

2 Open Interest Curve Across Strikes

Open interest in a call option denotes the total number of units in that contract that are held long at the close of the exchange; at the end of each trading day it is quoted for all call options that could be traded. Several plain vanilla options with various exercise prices and expiration months can be traded at any date on the same underlying security. In this paper we look at the dependence of open interest on the strike price for fixed maturity and underlying security. It is a stylized fact that a plot of that dependence for calls and puts shows a peak near to the at-the-money contract; this feature appears whatever the underlying stock, index or maturity. In this paper we focus on the series with the shortest maturity and call options.

An econometric analysis is beyond the scope of this paper. As an example we will present that feature for the options on Microsoft (ticker symbol: `MSFT`) and the NASDAQ 100 tracking unit (ticker symbol: `QQQ`) with April 2002 maturity. Expiration for that series was on Saturday April 20, 2002; we take a snapshot on the last four Fridays preceding that date.

The structure of the market in the third week of April 2002 is the following: with April 2002 maturity options on `MSFT` can be traded with strike prices in $5 unit intervals between $25 and $105 and those on `QQQ` with strike prices $1 unit intervals between $15 and $49. There are also contracts traded with maturity at the "Saturday immediately following the third Friday" in May 2002, June 2002, September 2002, December 2002, January 2003 and

January 2004 for QQQ and in May 2002, July 2002, October 2002, January 2003 and January 2004 for MSFT.

[Figure 1 about here.]

Figure 1 looks in the plots of left column at MSFT and in the right column at QQQ; rows one to four look seperately at Friday March 29, April 5, April 12 and April 19, 2002. The closing prices for MSFT (QQQ) on those four Fridays were \$59.44 (\$35.61), \$56.45 (\$34.76), \$54.79 (\$33), and \$56.37 (\$34.35), respectively[4]. They are indicated through a vertical line in each plot.

On all four Fridays and for both underlying securities figure 1 shows the stylized fact that there is a peak of open interest near to the at-the-money call option. Note that the stylized fact is evident whatever maturity we look at: it holds at very short horizons of one day up to four weeks. At those maturities risks are small; in the next section we look at allocations under small risk and expand in the risk parameter.

3 The Setup

We start with a zero-mean random variable z_0, numbers $K_1 < \ldots < K_N$ within the support of z_0 and a parameter $\varepsilon \geq 0$ that parameterizes a series of (financial) economies and corresponding portfolio problems. For technical reasons we assume here that z_0 is of bounded continuous support[5], i.e. there are numbers $z_{min} < z_{max}$ such that $\text{Prob}[z_0 \in [z_{min}, z_{max}]] = 1$. Each economy is populated by two agents that can invest today (date 0) into a riskless bond with constant price over time and $N+1$ risky securities[6]: a stock (security 0) and N call options with maturity at date 1 written on that stock. The distribution z_0 is known to both

[4]The grid is identical for calls and puts, i.e. for every call with a maturity T and a strike K also a put with same maturity and strike can be traded. Put-call parity allows replicating a long put position in a strike K through joint long positions in the stock, the call with strike K and an investment of $\$K$ in the bond. In this paper we focus on the equilibrium analysis of call options. Translating all positions into corresponding call positions and creating a new summary open interest in a call with strike K by adding to it the open interest in the put with strike K yields similar pictures.

[5]The boundedness condition ensures that stock payoffs are non-negative for a sufficiently small ε. The continuity assumption is made here for expositional reasons.

[6]We do not analyze the consumption-savings decision here and therefore set the riskless rate equal to 0 for simplicity.

agents. Agents do not trade between dates 0 and 1. At date 1 the payoff from the stock is $Y(\varepsilon) = 1 + z_0\varepsilon$ and that from the option $j = 1, \ldots, N$ is $\varepsilon(1 + z_0 - K_j)^+ = (Y(\varepsilon) - K_j(\varepsilon))^+$, where $K_j(\varepsilon) = 1 + \varepsilon(K_j - 1)$. Figure 2 depicts the payoff from option K in the ε-economy. Note that this is indeed a standard call option.

[Figure 2 about here.]

A characteristic feature of the expansion is that $\varepsilon = 0$ corresponds to an economy without risk: with probability 1 the payoff from the stock is identical to one ($Y(0) = 1$) and the payoff of all options vanishes. We change the strike price as ε changes such that the cumulative probability of exercise is unaffected by ε.

[Figure 3 about here.]

As an example figure 3 presents our expansion for parameters K_1, K_2, K_3 that are set at the first three quartiles, respectively: when $\varepsilon = 0$ the support of the distribution of $Y(0)$ is concentrated at exactly 1 and in our expansion all strikes collapse to exactly 1 so that none of them pays off. As ε increases, the support of $Y(\varepsilon)$ increases linearly around 1: For any $\varepsilon > 0$ the support is $[1 - \varepsilon z_{min}, 1 + \varepsilon z_{min}]$; this is characterized through the bold lines in that figure. As ε increases, the location of the quartiles increases linearly around 1 and to keep strikes at the quartiles we expand strikes linearly[7]. The location of the quartiles is is characterized through dashed lines in that figure.

We denote the date 0 price of securities by

$$P_0(\varepsilon) = E[Y(\varepsilon)] - \pi_0(\varepsilon) \cdot \varepsilon^2 \text{ for the stock,} \tag{1}$$

$$P_j(\varepsilon) = E[\varepsilon(1 + z_0 - K_j)^+] - \pi_j(\varepsilon) \cdot \varepsilon^2 \text{ for the options,} \tag{2}$$

and interpret $\pi_j(\varepsilon)\varepsilon^2$ as the *risk-premium* in the ε-economy. The variance of the stock and the option is $var(z_0)\varepsilon^2$ and $var((1 + z_0 - K_j)^+)\varepsilon^2$, respectively. When $\pi_j(\varepsilon)\varepsilon^2$ would be constant in ε this would model risk premia that are linear in variance; here we allow

[7] It is *not* feasible to keep strikes fixed since we are interested in the case where $\varepsilon \to 0$: for sufficiently small ε the probability of exercise would be zero and all options would vanish, because the support of z_0 is bounded.

$\pi_j(\varepsilon)$ to depend on ε to capture nonlinear dependence on the variance. Note that in the no-risk economy ($\varepsilon = 0$) the returns from the bond, stock and the call options coincide, i.e., $Y(0) = P_0(0)$ and the payoff and the price of the derivative vanish at $\varepsilon = 0$.

Each agent is endowed with 1/2 units of the stock, i.e. the total supply consists of one unit of stock, which is infinitely divisible. All call options are in zero net-supply. In each ε-economy agents pursue trading strategies over time: between 0 and 1 agent i invests $\$b_i(\varepsilon)$ of his initial wealth $W_{0i}(\varepsilon) = \frac{P_0(\varepsilon)}{2}$ into the riskless security and holds $d_{ij}(\varepsilon)$ units of risky security j $(0 \le j \le N)$, i.e. $W_{0i}(\varepsilon) = b_i(\varepsilon) + \sum_{j=0}^{N} d_{ij}(\varepsilon)P_j(\varepsilon)$. We re-express this as $b_i(\varepsilon) = W_{0i}(\varepsilon) - \sum_{j=0}^{N} d_j(\varepsilon)P_j(\varepsilon)$ and derive his total payoff (wealth) at date 1 as

$$W_{1i}(\varepsilon) = W_{0i}(\varepsilon) + d_{i0}(\varepsilon) \cdot (Y(\varepsilon) - P_0(\varepsilon)) + \sum_{j=1}^{N} d_{ij}(\varepsilon) \cdot (\varepsilon(1 + z_0 - K_j)^+ - P_j(\varepsilon)). \tag{3}$$

We assume agents do not consume today and that their date 1 preferences over wealth can be represented by von Neumann-Morgenstern utility functions $E[u_i(W_{1i}(\varepsilon))]$, where u_i is an increasing and strictly concave function. The agent then chooses the strategy that maximizes his expected utility $E\left[u_i\left(W_{1i}(\varepsilon)\right)\right]$ over all holdings $d_{ij}(\varepsilon)$ in the risky securities. Both agents trade competitively; we derive both agents' demand and prices in each security by the following equilibrium concept:

Definition 1 *A financial equilibrium in the ε-economy consists of prices $P_j(\varepsilon)$ for the available securities, and portfolio demand vectors $d_i(\varepsilon)$ for both agents, such that $d_i(\varepsilon)$ maximizes agent i's utility, fulfills equation (3), and stock and option markets clear, i.e. $d_{10}(\varepsilon)+d_{20}(\varepsilon) = 1$ and $d_{1j}(\varepsilon) + d_{2j}(\varepsilon) = 0$ $(j = 1, \dots, N)$.*

In each ε-economy open interest in an option $(j = 1, \dots, N)$ corresponds to the absolute value of either agent's demand in that option. Our setup will therefore provide us with a risk-sharing prediction of the open interest in options[8]. For each agent $i = 1, 2$ we define the $(N + 1)$-dimensional function $H_i(d_i(\varepsilon), \varepsilon)$ by

$$H_{ij}(d_i(\varepsilon), \varepsilon) = \frac{1}{\varepsilon} \cdot \frac{\partial E[u_i(W_{1i})]}{\partial d_{ij}} = E\left[\frac{\partial u_i}{\partial W}(W_{1i}(\varepsilon)) \cdot (z_j + \pi_j(\varepsilon)\varepsilon)\right]. \tag{4}$$

[8]Our model assumes that market participants evaluate the effects of holding options at a short horizon through buy-and-hold strategies. We believe the effects of retrading before maturity are small since we only look at very short horizons of one day up to maximally four weeks.

The *first order conditions*[9] of agent i in the ε economy are then $H_i(d_i(\varepsilon), \varepsilon) = 0$. To solve for individual demand and the equilibrium allocation we expand the demand vector, $d_i(\varepsilon)$, and the vector of premia, $\pi_j(\varepsilon)\varepsilon^2$, into series with respect to ε around $\varepsilon = 0$:

$$\begin{aligned} d_1(\varepsilon) &= d_1(0) + d_1'(0)\varepsilon + \ldots, d_2(\varepsilon) = d_2(0) + d_2'(0)\varepsilon + \ldots \\ \text{and } \pi(\varepsilon) &= \pi(0) + \pi'(0)\varepsilon + \ldots, \end{aligned}$$

where $\pi'(\varepsilon) = \frac{\partial \pi}{\partial \varepsilon}(\varepsilon)$, and $d_i'(\varepsilon) = \frac{\partial d_i}{\partial \varepsilon}(\varepsilon)$. In this paper we are interested in economies in which the time between dates 0 and 1 is very small. The parameter ε controls the risks (standard deviation) in the economies; therefore we look below at the case where ε is small and study then the leading terms in the expansion of the allocation.

Our idea is to derive agent's holdings through an application of the Implicit Function Theorem: taking derivatives H_i with respect to ε, (formally) we get $0 = \frac{\partial H_i}{\partial d_i} \cdot \frac{\partial d_i}{\partial \varepsilon} + \frac{\partial H_i}{\partial \varepsilon}$ and we would like to derive $\frac{\partial d_i}{\partial \varepsilon}$ by dividing $\frac{\partial H_i}{\partial \varepsilon}$ by $\frac{\partial H_i}{\partial d_i}$. Yet, the way our expansion is set up, we have $\frac{\partial H_i}{\partial d_i}(d_i(0), \varepsilon = 0) = 0$: in the $\varepsilon = 0$ economy all securities coincide and therefore their demand is indeterminate. (A proof is provided in the appendix.) In the spirit of Hospital's rule we "require" $\frac{\partial H_i}{\partial \varepsilon}(d_i(0), \varepsilon = 0)$ to be also equal to zero and use second order derivatives of H_i to express $\frac{\partial d_i}{\partial \varepsilon}$.

This approach has been introduced by Judd and Guu (2001) (see their Theorem 7); here we will apply their procedure and for completeness we quote their generalized Implicit Function Theorem as Theorem 5 in the appendix. We refer to this as the *small-noise expansion* technique; it is a perturbation technique that was first introduced by Samuelson (1970) (see also Merton and Samuelson (1974)). The technique used here is an extension of Samuelson's: While Samuelson's analysis is asymptotically valid only for the zero-order term, however, Judd and Guu (2001) developed this to a technique that is asymptotically valid for all terms in the polynomial expansion.

The first three derivatives of agent i's utility function are denoted $u_i' = \frac{\partial u_i}{\partial W}$, $u_i'' = \frac{\partial^2 u_i}{\partial W^2}$, and $u_i''' = \frac{\partial^3 u_i}{\partial W^3}$, which are all evaluated at the agent's "safe" wealth $W_{0i}(0) = \frac{P_0(0)}{2} = \frac{1}{2}$.

[9]To simplify the exposition we divided by ε when defining H_i in equation (4). A repeated application of our procedure would yield the same result, see Judd and Guu (2001).

Moreover, for agent i we define the *risk-tolerance* τ_i and the *skew-tolerance* ρ_i by

$$\tau_i = -\frac{u_i'}{u_i''}, \ \rho_i = \frac{\tau_i^2}{2}\frac{u_i'''}{u_i'} = \frac{1}{2}\frac{u_i'}{u_i''}\frac{u_i'''}{u_i''}. \tag{5}$$

It is important to note that the risk-tolerance τ_i consists of first and second order derivatives of agents' utility functions; it describes the marginal rate of substitution between mean and variance and is a usual term in the financial economics literature. The term skew-tolerance[10] appeared first in Judd and Guu (2001); additionally it contains a third order derivative of agent's utility function. We explain below that this term describes the marginal rate of substitution between skewness and variance risk.

We denote by $z_j = (1+z_0-K_j)^+ - E[(1+z_0-K_j)^+]$ the pure random component of options $j=1,\ldots,N$, by V the $(N+1)\times(N+1)$ dimensional *covariance matrix* of the stock and all options, by χ the $N+1$ dimensional *co-skewness vector*, and for agent $i=1,2$ by ζ_i the $N+1$ dimensional *(third order) co-moment vector*, i.e. for securities $j,k=0,...,N$,

$$V_{jk} = E[z_j z_k], \chi_j = E[z_0^2 z_j], \zeta_{ij} = \sum_{k,l=0}^{N} d_{ik}(0) d_{il}(0) E[z_j z_k z_l]. \tag{6}$$

Co-skewness is a known term in the financial economics literature, see Huang and Litzenberger (1988). It has recently found renewed interest in empirical studies by Harvey and Siddique (2000), Dittmar (2002), and Chang, Johnson and Schill (2002) who look at extensions of the CAPM framework to determine if the squared return on the market portfolio is priced[11]. We argue that a third moment leads to option demand and is priced.

Note that the variance-covariance matrix V of the $N+1$ securities is not singular, i.e. that V is invertible, since the support of z_0 is continuous.

[10] Kimball (1990) refers to $\frac{\rho_i}{\tau_i} = -\frac{u_i'''}{2u''}$ as prudence. Note that $\frac{\partial \tau_i}{\partial W} = \frac{-(u_i'')^2 + u_i' u_i'''}{(u_i'')^2} = -1 + 2\rho_i$. Typically it is assumed that the risk-tolerance τ_i is increasing in wealth; therefore we expect $\rho_i > 0$. We will not pursue these relations here further.

[11] Co-skewness is defined in that literature for a security as the covariance of the squared market risk with the idiosyncratic risk of that security. In our setup this definition of co-skewness and ours coincide. Our focus is on option demand whereas that literature looks at pricing implications; therefore we will not pursue the relationship between coskewness as defined here and the one defined typically in the literature.

4 Allocations in The Small-Noise Expansion

4.1 How to Invest into a Stock and a Portfolio of Calls

The appendix proves:

Theorem 2 *Agent i's demand vector* $d_i(\varepsilon) = d_i(0) + d_i'(0)\varepsilon + \dots$ *is described through*

$$d_i\,(0) = \tau_i \cdot V^{-1} \cdot \pi\,(0) \;\; \text{and} \;\; d_i'\,(0) = \tau_i \cdot V^{-1} \cdot \left(\pi'\,(0) + \frac{\rho_i}{\tau_i^2} \cdot \zeta_i\right). \tag{7}$$

We interpret the demand terms in the asymptotic expansion $d_i(\varepsilon) = d_i(0)+d_i'(0)\varepsilon+\mathcal{O}(\varepsilon^2)$ as follows: the $d_i(0)$ term consists of the premium, $\pi(0)$, standardized by the variance, V, of the stock risk; the agent's risk-tolerance τ_i describes the marginal rate of substitution between variance risk and the premium gained for taking that risk. This result is common to economies in which agents' preferences can be summarized in terms of means and variances, see, Huang and Litzenberger (1988).

The $d_i'\,(0)$ term takes into account all third-order cross moments via ζ_i. We rewrite demand using equation (7) as $d_i\,(\varepsilon) = \tau_i \cdot V^{-1} \cdot (\pi\,(0) + \pi'\,(0)\,\varepsilon) + \frac{\rho_i}{\tau_i^2} V^{-1}\zeta_i\varepsilon + \mathcal{O}\,(\varepsilon^2)$ and note that $\zeta_i = E[\eta_i^2 z_i] = Cov(\eta_i^2, z_i)$ corresponds to a linear regression of a security's risk z_i on agent i's (zero-order) wealth risk $\eta_i = \sum_{j=0}^{N} d_{ij}(0)z_j$. We interpret this as a correction to the agents risk coming from his zero-order position in securities (the zero order wealth risk) $\eta_i = \sum_{j=0}^{N} d_{ij}(0)z_j$. (Note that in the zero-order term the agent cares how his squared wealth risk covaries with security i, i.e. the $Cov(\eta_i, z_i)$ term determines the position in security j.) Equation (7) explains that the term $\frac{\rho_i}{\tau_i^2} = \frac{u_i'''}{2u_i'}$ characterizes the marginal rate of substitution between (third moment) risk ζ_i and the compensation (part of the premium) received for taking that kind or risk.

4.2 The Equilibrium Allocation

In equilibrium agents have to agree on a market clearing price in all ε-economies, i.e. $d_{10}(\varepsilon) + d_{20}(\varepsilon) = 1$ for the stock and $d_{1i}(\varepsilon) + d_{2i}(\varepsilon) = 0$ for the options ($j = 1, \dots, N$). For the zero and first order expansion terms of $d_i(\varepsilon)$ in our series expansion this translates into $d_{10}(0) + d_{20}(0) = 1$, $d_{10}'(0) + d_{20}'(0) = 0$ and $d_{1j}(0) + d_{2j}(0) = 0$, $d_{1j}'(0) + d_{2j}'(0) = 0$ for options $j = 1, \dots, N$. The appendix proves:

Theorem 3 *The premium of security $j = 0, \ldots, N$ is:*

$$\pi_j(\varepsilon)\varepsilon^2 = \frac{1}{\tau_1+\tau_2}V_{0j}\varepsilon^2 - \frac{\rho_1\tau_1+\rho_2\tau_2}{(\tau_1+\tau_2)^3}\chi_j\varepsilon^3 + \mathcal{O}\left(\varepsilon^4\right), \tag{8}$$

and the first agent's demand is:

$$\begin{aligned} d_{10}(\varepsilon) &= \frac{\tau_1}{\tau_1+\tau_2} - \tau_1\tau_2\frac{\rho_1-\rho_2}{(\tau_1+\tau_2)^3}\cdot\left(V^{-1}\chi\right)_0\varepsilon + \mathcal{O}\left(\varepsilon^2\right) \text{ for the stock, and} \\ d_{1j}(\varepsilon) &= -\tau_1\tau_2\frac{\rho_1-\rho_2}{(\tau_1+\tau_2)^3}\cdot\left(V^{-1}\chi\right)_j\varepsilon + \mathcal{O}\left(\varepsilon^2\right) \text{ for options } (j=1,\ldots,N). \end{aligned}$$

Note that $d_{ij}(0) = 0$ ($j = 1, \ldots, N$) and that agent's zero order wealth risk therefore simplifies to $\eta_i = d_{i0}(0)z_0$. This implies that $\zeta_i = Cov(\eta_i^2, z_i) = d_{i0}^2(0)\chi_i$. (Here the stock plays a distinct role since it is the only non-financial security, i.e. it is the only security in positive aggregate supply. This distinct role is responsible for the simplification of ζ_i.) In our asymptotic analysis we focus here on the first term that is not equal to zero; since $d_{ij}(0) = 0$ for options $j = 1, \ldots, N$, zero and first order terms need to be analyzed.

It is important to note that according to theorem 3 we have separation of tastes (risk-preferences) and distributional characteristics in $d_i(0)$ and $d_i'(0)$. In particular, in a first approximation, tastes (risk-preferences) enter only as a multiplicative factor for the demand in all contracts. This will play a crucial role in our analysis of the shape of the open interest curve across strikes.

Based on theorem 3 and equations (1, 2) we find that prices are equal to

$$\begin{aligned} P_0(\varepsilon) &= E[Y(\varepsilon)] - \frac{1}{\tau_1+\tau_2}V_{00}\varepsilon^2 + \frac{\rho_1\tau_1+\rho_2\tau_2}{(\tau_1+\tau_2)^3}\chi_0\varepsilon^3 + \mathcal{O}\left(\varepsilon^4\right), \\ P_j(\varepsilon) &= E[\varepsilon(1+z_0-K_j)^+] - \frac{1}{\tau_1+\tau_2}V_{0j}\varepsilon^2 + \frac{\rho_1\tau_1+\rho_2\tau_2}{(\tau_1+\tau_2)^3}\chi_j\varepsilon^3 + \mathcal{O}\left(\varepsilon^4\right), \end{aligned}$$

where $j = 1, \ldots, N$. Here the $\frac{1}{\tau_1+\tau_2}V_{0j}\varepsilon^2$ term is exactly the pricing term that would result in a CAPM world. In addition we have the $\pi_j'(0)\varepsilon^2 = \frac{\rho_1\tau_1+\rho_2\tau_2}{(\tau_1+\tau_2)^3}\chi_j\varepsilon^2$ term which depends on preferences $\tau_1, \tau_2, \rho_1, \rho_2$ and co-skewness χ_j. Therefore, in our models co-skewness risk is priced. This confirms from a theoretical perspective the empirical evidence that points to the importance of skewness risk for pricing: Kraus and Litzenberger (1976) considered moments higher than variance for agent's expected utility[12] and found empirically that only

[12]Kraus and Litzenberger (1976) perform a Taylor series expansion of agent's utility function that is truncated at the third moment. The small-noise expansion provides a theoretical basis for this ad-hoc procedure.

systematic skewness risk is priced. Recently Harvey and Siddique (2000), Dittmar (2002) and Chang, Johnson and Schill (2002) studied extensions of the basic CAPM setup and concluded that co-skewness risk is priced in the market in addition to variance risk.

4.3 Why Agents Trade Options

Theorem 3 states that for options the zero order equilibrium demand term vanishes. The zero-order term corresponds to a mean-variance framework and therefore our result is in line with common knowledge about such frameworks: it is known that two-fund separation rules hold, i.e. agents hold the bond and the market portfolio. Options are not contained in the "market" portfolio, since they are in zero net-supply. Therefore, in *our* economy the "market" portfolio consists of one unit of the stock only and the zero-order term does not give rise to option trade[13].

We can write $V^{-1} \cdot \chi = V^{-1} \cdot E[z_0^2 z_i] = E[z_0^2 \cdot (V^{-1}z)] = cov(z_0^2, V^{-1}z)$ and then demand in option $j = 1, \ldots, N$ as

$$d_{1j}(\varepsilon) \quad = \quad -\tau_1\tau_2 \frac{\rho_1 - \rho_2}{(\tau_1 + \tau_2)^3} E[z_0^2 \cdot (V^{-1}z)]_j \varepsilon + \mathcal{O}\left(\varepsilon^2\right). \tag{9}$$

In this equation the term $V^{-1}z$ describes the orthogonal decomposition (in variance-covariance terms) of risks of all contracts; this decomposition characterizes the contributions each contract makes for hedging purposes. Agents care about the covariance with wealth risk. We explained above that with respect to the first order demand term the agent cares about hedging his zero-order wealth risk: since zero order wealth risk is given through the stock risk he looks for that purpose at the covariance of these components with squared stock risk z_0 to determine the demand in each contract. Therefore these terms are driven by co-skewness risk χ, i.e. the presence of third-order moment risk.

Demand is scaled by taste parameters $-\tau_1\tau_2 \frac{\rho_1 - \rho_2}{(\tau_1+\tau_2)^3}$. Agents typically have elaborate preferences over risk, e.g. they care about asymmetric events like liquidity shocks and market crashes; we parameterize these as preferences over skewness and assume that neither ρ_1 nor ρ_2 are equal to zero.

[13]Similarly, Detemple and Selden (1991) first pointed out that introducing a new contract might not lead to trade.

Note that when both agents have identical skew-tolerance ($\rho_1 = \rho_2$), then no option demand arises in a first approximation in ε, i.e. $d_j'(0) = 0$ for $j = 1, \ldots, N$. (This captures, e.g. the case of identical agents in the economy.) However equilibrium demand in options will arise from skew-tolerance and co-skewness as long as $\rho_1 \neq \rho_2$. In general we expect $\rho_1 \neq \rho_2$ and find that this justifies equilibrium trade in options in our setup. The intuitive basis of our argument is that call options are contracts that allow trading of events that are in the upper tail of the stock distribution; trading options allows agents to alter the skewness of their portfolio payoff.

5 The Open Interest Curve In Incomplete Markets

Throughout this section we study the leading term in the expansion looking at various distributions of the underlying security where the support is continuous and z_0 is distributed over the interval $(-1, 1)$, i.e. $1+z_0\varepsilon$ is distributed over the interval $[z_{min}, z_{max}] = [1-\varepsilon, 1+\varepsilon]$. We look at strike grids that are equidistant and symmetric around the center, vary the number N of traded options and the parameter K_1, and denote the difference between adjacent strike parameters by $\delta = 2\frac{1-K_1}{N-1}$. K_1 and N control how dense the strike grid is and how far it spreads out within the support of the distribution. We allow only strikes within that support, since strikes outside the support of the distributions would violate the non-redundancy condition; this restricts the interval within which K_1 varies to $[0, 1]$.

5.1 The Leading Term in the Open Interest Curve With A Small Number of Options

We assume that the taste parameter $\rho_1 \neq \rho_2$ so that $\tau_1\tau_2\frac{\rho_1-\rho_2}{(\tau_1+\tau_2)^3} \neq 0$ and define the N-dimensional vector θ_j for $j = 1, \ldots, N$ by

$$\theta_j = \frac{d_{1j}'(0)}{\delta \cdot \tau_1\tau_2\frac{\rho_1-\rho_2}{(\tau_1+\tau_2)^3}}, \text{ so that } |d_{1j}(\varepsilon)| = \frac{|(V^{-1}\chi)_j|}{\delta} + \mathcal{O}(\varepsilon) = |\theta_j| + \mathcal{O}(\varepsilon). \tag{10}$$

This vector is a standardization of the leading term in the ε-expansion of open interest with respect to the taste parameter $\tau_1\tau_2\frac{\rho_1-\rho_2}{(\tau_1+\tau_2)^3}$ and the distance $\delta = 2\frac{1-K_1}{N-1}$ between adjacent strike parameters. Since open interest separates into tastes and preferences by theorem 3, we can standardize with respect to the taste parameter and so the leading terms in the relative size are determined through θ. The standardization with respect to δ is necessary when the number of traded options is increased. We are mostly interested in the case where ε is small; throughout this section we will therefore only analyze this term. We perform analyses with respect to three distributions:

1. Uniform distribution on the interval $(-1,1)$: We assume that z_0 is uniformly distributed over the interval $(-1,1)$; the density of z_0 is

$$1_{(-1,1)}(x) = \begin{cases} 1 & \text{if } -1 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases} .$$

2. Normal distribution truncated to the interval $(-1,1)$: We take $z_0 \sim \mathcal{N}(0, 1/3)$ conditional on z_0 in $(-1,1)$; the density of z_0 is

$$f(x) = 1_{(-1,1)}(x) \frac{1}{\sqrt{2 \cdot \pi \cdot 1/3}} \exp\left(-\frac{x^2}{2 \cdot 1/3}\right).$$

We find that the mean is 0 and the variance is 0.2215. (Note that the variance is *not* equal to 1/3 since we truncate the distribution.)

3. (Transformed) Lognormal truncated to the interval $[-1,1]$: $z_0 \sim \exp(X) - 1$ where $X \sim \mathcal{N}(0.036, 1/3)$, conditional on z_0 in $[-1,1]$; the density of z_0 is

$$f(x) = 1_{(-1,1)}(x) \frac{1}{(1+x)\sqrt{2 \cdot \pi \cdot 1/3}} \exp\left(-\frac{(\ln(1+x))^2}{2 \cdot 1/3}\right).$$

We find that the mean is 0 and the variance is 0.1840. (The variance here is *not* equal to 1/3 since we truncate the distribution.)

Straightforward calculations reveal that for this distribution the mean is $E[1+z_0] = 1$, the variance is $Var(1+z_0) = \frac{1}{3}$, that $E[(1+z_0-K_j)^+] = \frac{(K_j-2)^2}{4}$, $E[(1+z_0)\cdot(1+z_0-K_j)] = \frac{(K_j-2)^2(K_j-4)}{12}$, $E[((1+z_0-K_j)^+)^2] = -\frac{(K_j-2)^3}{6}$ and for $K_j < K_k$ that $E[(1+z_0-K_j)^+ \cdot (1+z_0-K_k)^+] = -\frac{(K_k-2)^2(3K_j-K_k-4)}{12}$. Also we calculate $E[(1+z_0)^2 \cdot (1+z_0-K_j)^+] = \frac{(K_j^2+2)(-2+K_j)^2}{24}$. This implies that co-variance terms are $V_{00} = \frac{1}{3}$ for the stock, and for options $j,k = 1,\ldots,N$: $V_{0j} = \frac{(K_j+1)(-2+K_j)^2}{12}$, $V_{jj} = -\frac{(3K_j+2)(-2+K_j)^3}{48}$, and $V_{jk} = -\frac{(-2+K_k)^2(-4K_k-4+3K_j^2)}{48}$. Skewness is $\chi_0 = 0$, $\chi_j = \frac{K_j^2(-2+K_j)^2}{24}$.

We will now discuss a case in which 3 options are traded and one in which 4 options are traded. With three options we get:

$$\theta = \begin{pmatrix} \frac{1}{(K_1-1)^2(2K_1^2-5K_1-1)} \\ 2\frac{(2K_1^2-7K_1+4)K_1}{(K_1-1)^2(2K_1^2-5K_1-1)} \\ \frac{1}{(K_1-1)^2(2K_1^2-5K_1-1)} \end{pmatrix}, \tag{11}$$

where θ is as defined in equation (10). (Note that this standardized demand is symmetric, since the uniform distribution is symmetric.) In a first order approximation in ε a peak in the open interest curve results if and only if $|\theta_1| < |\theta_2|$.

A straightforward analysis of maxima based on equation (11) proves that $|\theta_1| < |\theta_2|$, if and only if either $0.1771 < K_1 < 0.5$ or $0.8696 < K_1 < 1$. Therefore we need to distinguish between four subintervals of the interval $[0, 1]$ with cutoff points $0 < 0.1771 < 0.5 < 0.8696 < 1$. When K_1 varies within two out of these four subintervals the $|d'_{ij}(0)|$ terms peak for the at-the-money option; when ε is small this would support the stylized fact. However, for the other two out of four subintervals it does not support the stylized fact: we get a dip at-the-money. On the intervals where a peak occurs, the relative size of that peak is not what we see in practice. Taking the limit[14] $K_1 \to 1$ we get $\frac{|\theta_2|}{|\theta_1|} \to 2$ and while this indicates a peak at-the-money, the size is too small. Next we look at the behavior where K_1 is neither close to 0 nor close to 1 by looking at the ratio $\frac{|\theta_2|}{|\theta_1|}$ as a function of K_1: on the interval $[0, 0.5]$ the maximum of this ratio is 1.2593 at $K_1 = 1/3$. Again, this size is too small for practical relevance.

With 4 options we get:

$$\theta = \begin{pmatrix} -3\frac{(K_1^2-2K_1-2)(2+K_1)^2}{4(-1+K_1)^2(-4-60K_1+15K_1^2+4K_1^3)} \\ 9\frac{(3K_1^3+8K_1^2-52K_1+32)K_1}{4(-1+K_1)^2(-4-60K_1+15K_1^2+4K_1^3)} \\ 9\frac{(3K_1^3+8K_1^2-52K_1+32)K_1}{4(-1+K_1)^2(-4-60K_1+15K_1^2+4K_1^3)} \\ -3\frac{(K_1^2-2K_1-2)(2+K_1)^2}{4(-1+K_1)^2(-4-60K_1+15K_1^2+4K_1^3)} \end{pmatrix}. \tag{12}$$

In a first order approximation in ε a peak in the open interest curve is characterized by $|\theta_1| < |\theta_2|$. A straightforward analysis of maxima based on equation (12) yields that

[14] The other limiting case where K_1 tends to the boundaries of the support of the distribution is $K_1 \to 0$; we ignore this here since have seen above that close to 0 we get a dip at-the-money. Both limiting case are abnormal, since for $K_1 \to 1$ all strikes "collapse" to a single one, i.e. all options become more and more "similar," and therefore agents try to "leverage" their position by holding a long position in one of them and a short in the other and we find that $|\theta_1|, |\theta_2|, |\theta_3| \to \infty$.

$|\theta_1| < |\theta_2|$ if and only if $0.1372 < K_1 < 0.4$. Therefore we need to distinguish between three subintervals with cutoff points $0 < 0.1372 < 0.4 < K_1$. When K_1 varies within one out of these three subintervals the $d'_{ij}(0)$ term peaks for the at-the-money option; when ε is small this would support the stylized fact. However, for the other two out of the three subintervals our setup does not support the stylized fact. (We will not analyze any of the limits $K_1 \to 0$ or $K_1 \to 1$ since on those interval a dip results.)

So far we analyzed the open interest curve for the uniform distribution, only. We found that the shape of the open interest curve is very sensitive to the number of options traded and the location of the strike grid: depending on what these parameters are we found a peak and a dip.

[Figure 4 about here.]

We now plot θ, the leading term in standardized open interest, for the uniform, the (truncated) normal and the (truncated) lognormal distributions. We calculate approximations of the co-variance and skewness terms through a numerical integration scheme with 20000 nodes on the interval $(-1, 1)$, i.e. the step size is 0.0001. Therefore we expect our results to be accurate at four digits. We illustrate in figure 4 the open interest curve when three options are traded: they differ in the type of distribution (columns one to three are the uniform, normal and the lognormal, respectively) and in the minimal strike (row one $K_1 = 0.1$) and row two $K_1 = 0.3$). The figure confirms what we have seen above in our analysis of the equations for the uniform distribution: we find a peak and a dip and the presence of either of these shapes is very sensitive to the strike grid. We also find that the shape of the curve is very sensitive to the distributional assumption and that there is at first sight no commonality between the plots; therefore we conclude that the shape is also very sensitive to the distribution.

The leading term in the curve leads to a variety of different shapes; it determines the shape of the open interest curve in incomplete markets with a small number of options. Overall we therefore conclude that in such markets the open interest curve is very sensitive to the number of options traded, to the type of distribution and the location of the strike grid.

5.2 The Leading Term in the Open Interest Curve With a Large Number of Options

We will now analyze the vector θ when z_0 is the uniform distribution and $N = 10$ options can be traded. In this case option demand is symmetric around the center strike and we have $\theta_6'(0) = \theta_5'(0)$, $\theta_7'(0) = \theta_4'(0)$, $\theta_8'(0) = \theta_3'(0)$, $\theta_9'(0) = \theta_2'(0)$, $\theta_{10}'(0) = \theta_1'(0)$. We calculate that

$$\begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_5 \end{pmatrix} = \begin{pmatrix} \frac{1}{18}\frac{(17+119K_1+116K_1^2)(2+7K_1)^2}{(-1+K_1)^2K_1(68+197K_1)} \\ -\frac{1}{4}\frac{(-544-772K_1+3176K_1^2+3891K_1^3)}{(-1+K_1)^2(68+197K_1)} \\ \frac{1}{4}\frac{(544+412K_1-2456K_1^2+3039K_1^3)}{(-1+K_1)^2(68+197K_1)} \\ \frac{1}{4}\frac{(544+508K_1-2648K_1^2+1191K_1^3)}{(-1+K_1)^2(68+197K_1)} \\ \frac{1}{4}\frac{(544+484K_1-2600K_1^2+1653K_1^3)}{(-1+K_1)^2(68+197K_1)} \end{pmatrix}.$$

On the interval $[0,1]$ we define the function $\varphi(K_1) = K_1\frac{-4+8K_1+77K_1^2}{(-1+K_1)^2(68+197K_1)}$. It is a straightforward linear manipulation to check that for the inner options differences in θ are described through the function φ as

$$\theta_2 - \theta_5 \quad = \quad -18\cdot\varphi(K_1), \theta_3 - \theta_5 = \frac{9}{2}\cdot\varphi(K_1), \theta_4 - \theta_5 = -\frac{3}{2}\cdot\varphi(K_1).$$

For the options with minimal and maximal strike parameters (K_1 and K_{10}) that difference is $\theta_1 - \theta_5 = -\frac{1}{36}\frac{(11K_1-2)(319K_1^3-3720K_1^2-1122K_1+68)}{(-1+K_1)^2K_1(68+197K_1)}$ and can not be re-expressed in simple terms of φ. Note that the extreme strikes serve a special purpose as they need to cover the remaining probability mass; we focus on the center behavior, i.e. on options $i = 2,\ldots,9$ and will not investigate these strikes further.

The function φ describes variations across θ as we change the minimal strike parameter. It has the following characteristics: the function φ is monotonically increasing on the interval 0 to 1/9, has a maximum with $\varphi(1/9) = 0.003379944$ at $K_1 = 1/9$, is monotonically decreasing on the interval 1/9 to 1 and has a singularity at 1, i.e. $\varphi(1) = -\infty$. We have $\varphi(0) = 0$ and on the interval $(0,1)$ the function φ has a single zero at $K_1 = 2/11$.

The parameter $K_1 = 2/11$ corresponds to the case where the strikes are equidistant within the support of the distribution, i.e. $K_1 = K_{j+1} - K_j = 1 - K_N$ for all $j = 1,\ldots,N$. We find in this special case that all $\theta_j = 2$ (for $j = 2,\ldots,9$), so that the contribution of the first-order expansion term to the shape of the open interest curve is flat across strikes (with the exception of the two extreme strikes).

When K_1 varies between 0 and $2/11 \approx 0.1818$ we find that all the θ terms are close to 2 (up to 2 digits). Therefore in plots the contribution of the first-order expansion term to the shape of the open interest curve would appear as flat for all these parameters.

[Figure 5 about here.]

We will now analyze the same distributions we looked at before and plot the vector θ for large numbers of traded options by varying the number of options and setting K_1 such that the grid is equidistant within the support of the distribution, i.e. such that $K_1 = K_{j+1} - K_j$. Figure 5 looks at $N = 15$ in the first row and $N = 25$ options in the second row for the uniform, truncated normal and truncated lognormal distributions (columns 1 to 3). We see in figure 5 that at the extremes a slight peak or a slight dip may occur. We ignore θ at the four most extreme strike parameters, i.e. for the two with the smallest strikes and the two with the largest strikes; then θ appears to be flat across strikes. It also appears as if that parameter is fairly independent of the type of distribution when N is "large" and that the higher the number of traded options the smaller becomes the difference in θ across strikes, even at the extremes.

Based on this we conclude that in incomplete markets with a large number of traded options higher order expansion terms need to be considered to determine the shape of the open interest curve. For analytic tractability we will perform that kind of analysis directly in complete markets case. This will be done in the following section.

6 The Open Interest Curve In Complete Markets

In this section we look at the case where the market is complete. Our setup is that of section 3 with the exception that options K can be traded for all K in the support of the distribution z_0; we refer to demand of that option by $d_{i,K}(\varepsilon)$ and assume that it pays $\varepsilon(1 + z_0 - K)^+$.

6.1 Structural Analysis of Open Interest Across Strikes

Breeden and Litzenberger (1978), Green and Jarrow (1987) and Nachman (1988) established that in complete markets any smooth function $h_i(z_0, \varepsilon)$ at date 1 can be attained by taking buy-and-hold positions in the available securities. We assume here that agent i maximizes expected utility $E[u_i(X_i(\varepsilon))]$, over all (measurable) functions $h_i(z_0, \varepsilon)$ that fulfill today's budget constraint $E\left[\phi(z_0, \varepsilon) \cdot \frac{Y(\varepsilon)}{2}\right] = W_{0i}(\varepsilon) = E[\phi(z_0, \varepsilon) \cdot h_i(z_0, \varepsilon)]$, where $\phi(z_0, \varepsilon)$ denotes the state-price density. The equilibrium concept then reads:

Definition 4 *A financial equilibrium in the ε-economy consists of a state-price density $\phi(z_0, \varepsilon)$ and smooth functions $h_i(z_0, \varepsilon)$ that maximize each agent's utility subject to his budget constraint such that markets clear, i.e. $h_1(z_0, \varepsilon) + h_2(z_0, \varepsilon) = Y(\varepsilon) = 1 + z_0\varepsilon$ almost surely.*

We denote by $S(Y(\varepsilon), \lambda(\varepsilon))$ the equilibrium payoff of agent 1 at date 1 and refer to this as the sharing function. Carr and Madan (2001) and Weinbaum (2001) explained how to take a static position in call options in order to attain a particular position at date 1. Their result implies with respect to the sharing function that

$$\begin{aligned} S(Y(\varepsilon), \lambda(\varepsilon)) &= b_i(\varepsilon) + d_{1,0}(\varepsilon)Y(\varepsilon) + \int_0^\infty d_{1,K}(\varepsilon) \cdot \varepsilon \cdot (1 + z_0 - K)^+ dK \\ \text{where } d_{1,K}(\varepsilon) &= \varepsilon \cdot S''(K, \lambda(\varepsilon)). \end{aligned}$$

(As before $b_i(\varepsilon)$ and $d_{1,0}(\varepsilon)$ denote the bond and stock holdings of the agent i.) The second derivative here is taken with respect to the Y entry. Note that the optimal payoff of agent 2 is then $Y(\varepsilon) - S(Y(\varepsilon), \lambda(\varepsilon))$ and so $d_{1,K}(\varepsilon) = \varepsilon \cdot \frac{\partial^2 (Y(\varepsilon) - S(Y(\varepsilon), \lambda(\varepsilon)))}{\partial Y^2}_{|Y=K} = \varepsilon \cdot S''(K, \lambda(\varepsilon)) = -d_{1,K}(\varepsilon)$. It is known that the equilibrium is Pareto-optimal, i.e.

$$0 = u_1'(S(Y)) - \lambda(\varepsilon) \cdot u_2'(Y - S(Y)), \tag{13}$$

where the parameter $\lambda(\varepsilon) = \frac{u_1'(P_0(\varepsilon)/2)}{u_2'(P_0(\varepsilon)/2)}$ characterizes income effects. We denote

$$\kappa_i = \frac{\tau_i^3}{3} \frac{u_i''''}{u_i'}, \xi_i = \frac{\tau_i^4}{4} \frac{u_i^{(5)}}{u_i'}, \tag{14}$$

where the derivatives of agent's utility function are evaluated at their wealth in the no-risk economy, i.e. at $\frac{P_0(0)}{2} = \frac{1}{2}$ and continue to use τ_i, ρ_i as defined in equation (5). Since

$S(1) = \frac{P_0(0)}{2} = \frac{1}{2}$, repeated derivatives of equation (13) with respect to Y yield

$$S'(1) = \frac{\tau_1}{\tau_1 + \tau_2}, \; S''(1) = \tau_1 \tau_2 \frac{\rho_1 - \rho_2}{(\tau_1 + \tau_2)^3}, \tag{15}$$

$$S'''(1) = 3\tau_1\tau_2 \frac{\kappa_1 - \kappa_2}{(\tau_1 + \tau_2)^4} + 12\tau_1\tau_2(\tau_1\rho_2 + \tau_2\rho_1)\frac{\rho_1 - \rho_2}{(\tau_1 + \tau_2)^5} \tag{16}$$

$$S^{(4)}(1) = 4\tau_1\tau_2 \frac{\xi_1 - \xi_2}{(\tau_1 + \tau_2)^5} + 36\tau_1\tau_2(\rho_1 - \rho_2)\frac{\tau_1\kappa_2 + \tau_2\kappa_1}{(\tau_1 + \tau_2)^6} \tag{17}$$

$$+24\tau_1\tau_2(\rho_1 - \rho_2)^2 \frac{\rho_1\tau_2^2 - \rho_2\tau_1^2}{(\tau_1 + \tau_2)^7} + 24\tau_1\tau_2(\kappa_1 - \kappa_2)(\rho_1 - \rho_2)^2 \frac{\rho_1\tau_2 + \rho_2\tau_1}{(\tau_1 + \tau_2)^6} \tag{18}$$

$$+96\tau_1\tau_2(\rho_1 - \rho_2)^3 \frac{(\rho_1\tau_2 + \rho_2\tau_1)^2}{(\tau_1 + \tau_2)^7} \tag{19}$$

In general income effects at date 0 influence the optimal sharing function. That general case will be discussed at the end of this subsection and we will explain that these effects are of second order importance. A full analysis of λ has not been done previously in the literature and is also beyond the scope of our paper. Except otherwise explicitly noted we will therefore ignore income effects and assume that λ is constant in ε throughout the remainder of this section. Under that assumption the optimal sharing rule S depends on ε only via $Y(\varepsilon)$; a Taylor series expansion of $\varepsilon S''(Y)$ around 1 then gives:

$$d_{i,K}(\varepsilon) = \varepsilon S''(K) = \varepsilon S''(1) + K\varepsilon^2 S'''(1) + \frac{1}{2}K^2\varepsilon^3 S''''(1) + \frac{1}{6}K^3\varepsilon^4 S'''''(1) + \mathcal{O}(\varepsilon^5). \tag{20}$$

Note that the $S^{(j)}(1)$ terms are constants here. For fixed ε we are interested in determining the effects of the various terms in that equation on the shape of the open interest curve. The first term in equation (20) determines the level of option demand; it is flat across strikes with a value of $\varepsilon S''(1)$. The term of next higher order adds a linear term of slope $\varepsilon^2 S'''(1)$ and the following term a quadratic term $\varepsilon^3 S''''(1)$ into the curve. Higher order terms in ε introduce terms of third, fourth and higher order in K. The first three terms in this expansion are the necessary leading terms to determine the shape of the open interest curve up to curvature terms. Therefore we will now study these in more detail.

Before doing so let us compare the result of the incomplete markets case of subsection 5.2 with the result here that the leading term in the complete markets case is flat with a value of $\varepsilon S''(1) = \tau_1\tau_2 \frac{\rho_1 - \rho_2}{(\tau_1+\tau_2)^3}$: We denote by $K_{n,j} = z_{min} + \frac{z_{max} - z_{min}}{n+2}$ for $n = 0, 1, 2, \ldots$ and $j = 1, \ldots, n-1$ equidistant numbers within the support of z_0. Therefore $\sum_{j=1}^{N} d'_{i,K_{n,j}}(0)(1 +$

$z_0 - K_j)^+ = \sum_{j=1}^{N} \frac{1}{\delta} d'_{i,K_{n,j}}(0) \cdot (1 + z_0 - K_j)^+ \delta \to \int_0^\infty 2\tau_1\tau_2 \frac{\rho_1 - \rho_2}{(\tau_1+\tau_2)^3}(1 + z_0 - K_j)^+ dK$. Note that when we set $K_1 = \delta = \frac{z_{max} - z_{min}}{n}$ and take the set $K_{n,j}$ $(j = 1, \ldots, n-1)$ to determine the strikes in subsection 5.2 this result suggests that in that section $\theta_j \approx 2$. However this is exactly what we have seen in the simulations done there. This validates our analysis of subsection 5.2.

In the general case where income effects influence S, a two-dimensional Taylor-series expansion of S yields that $d_{i,K}(\varepsilon)$ is equal to

$$\begin{aligned}
&\varepsilon S^{(2,0)} + \varepsilon^2 \left(\lambda' S^{(2,1)} + K S^{(3,0)}\right) + \varepsilon^3 \left(\frac{1}{2}\lambda'' S^{(2,1)} + \frac{1}{2}(\lambda')^2 S^{(2,2)} + K\lambda' S^{(3,1)} + \frac{1}{2}K^2 S^{(4,0)}\right) \\
&+\varepsilon^4 \Bigg\{ \frac{1}{6}\lambda''' S^{(2,1)} + \frac{1}{2}\lambda'\lambda'' S^{(2,2)} + \frac{1}{6}(\lambda')^3 S^{(2,3)} + K\left(\frac{1}{2}\lambda'' S^{(3,1)} + \frac{1}{2}(\lambda')^2 S^{(3,2)}\right) \\
&\quad + \frac{1}{2}K^2\lambda' S^{(4,1)} + \frac{1}{6}K^3 S^{(5,0)} \Bigg\} + \mathcal{O}(\varepsilon^5),
\end{aligned}$$

where $S^{(j,k)}$ denotes the j-th derivative with respect to the first entry and the k-th derivative with respect to the second entry. (The function λ is here evaluated at $\varepsilon = 0$ and the function S at $Y(0) = 1$; we dropped them here to simplify the presentation.)

Furthermore we find that the $S^{(j,0)}(1)$ terms are equal to the $S^{(j)}(1)$ terms in equations (15-19), where we ignored income effects. In this equation the level is set by $\varepsilon S^{(2,0)}$ as in the special case where we ignored income effects. The second, third, and fourth order term in ε differ each between equation (20) and the one here, but both introduce a linear, quadratic and cubic term, respectively into the analysis.

The first, second and third order terms are here the leading terms to perform an analysis of the shape of the open interest curve. Our analysis here confirm that the first three leading terms need to be studied to determine the shape of the open interest curve.

6.2 The Shape of The Open Interest Curve With HARA utility

As an example to illustrate the sensitivity of the open interest curve to assumptions about preferences we look at two special cases of HARA utility.

When the first agent has a logarithmic utility function $(u_1(w) = \log(w))$ and the second agent has a square-root utility function $(u_2(w) = \sqrt{w}\ )$ Wang (1996) calculated the sharing function in a one-dimensional diffusion setup[15]:

$$S(Y(\varepsilon), \lambda(\varepsilon)) = \frac{2}{\lambda^2(\varepsilon)} \left\{ \sqrt{1 + 4\lambda^2(\varepsilon) Y(\varepsilon)} - 1 \right\}. \tag{21}$$

This a special case for which the solution to the open interest curve can be calculated in closed-form[16]. We also calculate $\lambda(0) = \frac{u_1'(1/2)}{u_2'(1/2)} = \sqrt{8}$. The second derivative determines option demand and so

$$S''(Y(\varepsilon)) = -\frac{4}{\sqrt{1 + 8Y(\varepsilon)}}. \tag{22}$$

[Figure 6 about here.]

Figure 6 depicts that open interest curve. We have also depicted the range of potential strike prices when the time increment between dates 0 and 1 is one week and when it is one month. (We choose ε to be the time increment here.) Our assumption here is that the annualized volatility is 60%; this matches approximately the volatility of `MSFT`. We truncated the distribution at 3 standard deviations, i.e. we set $z_{min} = -3 \cdot 60\%, z_{max} = 3 \cdot 60\%$. For a normal distribution this would cover 99.7% of the probability mass and we expect it to be a reasonable approximation for other distributions, too.

We find that the open interest curve is monotonically decreasing and of convex shape; it clearly does not support the stylized fact.

[Figure 7 about here.]

We will now use that example to address the accuracy of our approximation. Figure 7 plots the relative errors of our approximation to that of the closed-form solution in equation (22) using the first, second and third order approximation, i.e. $\frac{\varepsilon S''(1) - d_{i,K}(\varepsilon)}{d_{i,K}(\varepsilon)}$, $\frac{\varepsilon S''(1) + \varepsilon^2 K S'''(1) - d_{i,K}(\varepsilon)}{d_{i,K}(\varepsilon)}$, and $\frac{\varepsilon S''(1) + \varepsilon^2 K S'''(1) + 0.5\varepsilon^3 K^2 S'''(1) - d_{i,K}(\varepsilon)}{d_{i,K}(\varepsilon)}$, respectively. (Note that in equation (22) $d_{i,K}(\varepsilon) > 0$ for all strike parameters.)

[15] This result depends critically on the fact that both agents have CRRA utility where the coefficient of one agent is twice that of the other agent. To our knowledge these are the only cases for which the sharing function is known in closed-form; all these cases lead to similar results and we focus here on the original one of Wang (1996). Note that the sharing rule has this structure, independent of whether agents can retrade or not between dates 0 and 1.

[16] We are grateful to Jun Pan for pointing out this example to us.

In that figure we depict the range for horizons of one week and one month as explained after figure 6. We find that towards the extremes the accuracy of all approximation terms is decreasing. The approximation of highest order is the most accurate and leads to maximal relative errors of $\pm 20\%$ for a one month horizon and less than $\pm 5\%$ for a one week horizon. This term is therefore a good approximation and confirms us to look at this term throughout further analysis.

[Figure 8 about here.]

Finally we look at the case where the first agent has logarithmic utility $u_1(w) = \log w$ and the second agent has exponential utility with parameter γ, $u_2(w) = -\exp(-\gamma w)$ and plot in figure 8 our third order approximation for the three parameters of γ: 0.9; 1; and 1.1. We find that the curve is convex and shows a peak for the at-the-money contract.

The link between the parameterization of the utility functions and the shape of the open interest curve could easily be studied in more detail since the leading terms are known in closed-form. Furthermore within our framework other utility functions could be incorporated. We refrain here from doing so since our purpose is to point out that a variety of shapes is possible within a narrow set of utility functions.

7 Conclusion

This paper derived the leading terms in an asymptotic expansion that describes agents' equilibrium demand schedules. We discussed that in incomplete markets with a small number of traded options the leading term in the expansion is very sensitive to assumptions about agent's preferences and the underlying distribution. When the number of options is increased we documented that the contribution of that term becomes more flat across strikes. We therefore performed an analysis of the necessary higher order terms that determine the shape in complete markets and found that in these markets the shape of the curve is sensitive to assumptions about agent's risk-preferences.

Appendix

A Perturbation Analysis

We prove in this appendix theorem 2 for any agent $i = 1, 2$. Since $W_{0i}(\varepsilon) = \frac{1}{2}P_0(\varepsilon) = \frac{1}{2}(1 - \pi_0(\varepsilon)\varepsilon^2)$ and therefore the wealth dynamics is

$$\begin{aligned}
W_{1i}(\varepsilon) &= \frac{1}{2}(1 - \pi_0(\varepsilon)\varepsilon^2) + \sum_{j=0}^{N} d_{ij}(\varepsilon) \cdot \left(z_j\varepsilon + \pi_j(\varepsilon) \cdot \varepsilon^2\right) \\
\frac{\partial W_{1i}}{\partial \varepsilon} &= -\frac{1}{2}(\pi_0'(\varepsilon)\varepsilon^2 + \pi_0(\varepsilon)2\varepsilon) + \sum_{j=0}^{N} d_{ij}(\varepsilon) \cdot \left(z_j + \pi_j'(\varepsilon) \cdot \varepsilon^2 + \pi_j'(\varepsilon)2\varepsilon\right) \\
\frac{\partial^2 W_{1i}}{\partial \varepsilon^2} &= -\frac{1}{2}(\pi_0''(\varepsilon)\varepsilon^2 + \pi_0'(\varepsilon)4\varepsilon + \pi_0(\varepsilon)2) + \sum_{j=0}^{N} d_{ij}(\varepsilon) \cdot \left(\pi_j''(\varepsilon)\varepsilon^2 + \pi_j'(\varepsilon)4\varepsilon + \pi_j'(\varepsilon)2\right).
\end{aligned}$$

We denote $\eta_i = \sum_{j=0}^{N} d_{ij}(0)z_j$, and find that $\eta_i = \frac{\partial W_{1i}}{\partial \varepsilon}(0)$ and $\frac{\partial^2 W_{1i}}{\partial \varepsilon^2}(0) = -\pi_0(0) + \sum_{j=0}^{N} d_{ij}(0)2\pi_j(0)$. Note that $E[\eta_i] = 0$ and $\zeta_{ij} = E\left[\left(\sum_{k=0}^{N} d_{ik}(0)z_k\right)^2 \cdot z_j\right] = E[\eta_i^2 z_j]$.

Theorem 5 *(Theorem 7 in Judd and Guu (2001)) Suppose $H : R^n \times R \to R^n$ is analytic, and $H(x, 0) = 0$ for all $x \in R^n$. Furthermore, suppose that for some $(x_0, 0)$*

$$\frac{\partial H}{\partial x}(x_0, 0) = 0_{n\times n}, \frac{\partial H}{\partial \varepsilon}(x_0, 0) = 0_n, \text{ and det } \left(\frac{\partial^2 H}{\partial x \partial \varepsilon}(x_0, 0)\right) \neq 0$$

Then there is an open neighborhood $\mathcal{N}$ of $(x_0, 0)$, and a function $h(\varepsilon) : R \to R^n, h(\varepsilon) \neq 0$ for $\varepsilon \neq 0$, such that

$$H(h(\varepsilon), \varepsilon) = 0 \text{ for } (h(\varepsilon), \varepsilon) \in \mathcal{N}$$

Furthermore, h is analytic and can be approximated by a Taylor series. In particular, the first order derivatives equal

$$h'(0) = -\frac{1}{2} \cdot \left(\frac{\partial^2 H}{\partial x \partial \varepsilon}\right)^{-1} \cdot \frac{\partial^2 H}{\partial \varepsilon^2}.$$

A.1 Calculating Derivatives

To apply theorem 5 we will now calculate the (first order) derivatives of H_i with respect to d and ε and then the (second order) derivatives with respect to (d, ε) and $(\varepsilon, \varepsilon)$:

$$\begin{aligned}
\frac{\partial H_{ij}}{\partial d_{ik}} &= E\left[\frac{\partial^2 u_i}{\partial W^2}(W_{1i}(\varepsilon)) \cdot (z_j + \pi_j(\varepsilon)\varepsilon) \cdot \left(z_k\varepsilon + \pi_k(\varepsilon)\varepsilon^2\right)\right] \\
\frac{\partial H_{ij}}{\partial \varepsilon} &= E\left[\frac{\partial^2 u}{\partial W^2}(W_{1i}(\varepsilon)) \cdot \frac{\partial W_{1i}}{\partial \varepsilon} \cdot (z_j + \pi_j(\varepsilon) \cdot \varepsilon) + \frac{\partial u_i}{\partial W}(W_{1i}(\varepsilon)) \cdot \left(\pi_j(\varepsilon) + \pi_j'(\varepsilon)\varepsilon\right)\right].
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial^2 H_{ij}}{\partial d_{ik}\partial \varepsilon} &= E\left[\frac{\partial^3 u_i}{\partial W^3}(W_{1i}(\varepsilon)) \cdot \frac{\partial W_{1i}}{\partial \varepsilon} \cdot (z_j + \pi_j(\varepsilon)\varepsilon)\left(z_k\varepsilon + \pi_k(\varepsilon)\varepsilon^2\right)\right. \\
&\quad + \frac{\partial^2 u_i}{\partial W_{1i}^2}(W_{1i}(\varepsilon))\left(\pi_j(\varepsilon) + \pi_j'(\varepsilon)\varepsilon\right)\left(z_k\varepsilon + \pi_k(\varepsilon)\varepsilon^2\right) \\
&\quad \left. + \frac{\partial^2 u_i}{\partial W^2}(W_{1i}(\varepsilon))(z_j + \pi_j(\varepsilon)\varepsilon^2)\left(z_k + \pi_k'(\varepsilon)\varepsilon^2 + \pi_k(\varepsilon)2\varepsilon\right)\right].
\end{aligned}$$

We also calculate

$$\begin{aligned}
\frac{\partial^2 H_{ij}}{\partial \varepsilon^2} &= E\left[\frac{\partial^3 u_i}{\partial W^3}(W_{1i}(\varepsilon)) \cdot \left(\frac{\partial W_{1i}}{\partial \varepsilon}\right)^2 \cdot (z_j + \pi_j(\varepsilon) \cdot \varepsilon)\right. \\
&\quad + \frac{\partial^2 u_i}{\partial W^2}(W_{1i}(\varepsilon)) \cdot \frac{\partial W_{1i}}{\partial \varepsilon} \cdot \left(\pi_j(\varepsilon) + \pi_j'(\varepsilon)\varepsilon\right) \\
&\quad \left. + \frac{\partial^2 u_i}{\partial W^2}(W_{1i}(\varepsilon)) \cdot \frac{\partial W_{1i}}{\partial \varepsilon} \cdot (z_j + \pi_j(\varepsilon)) + \frac{\partial u_i}{\partial W}(W_{1i}(\varepsilon)) \cdot \left(2\pi_j'(\varepsilon) + \pi_j''(\varepsilon)\varepsilon\right)\right].
\end{aligned}$$

A.2 Deriving The Zero and First Order Terms

We see that $\frac{\partial H_{ij}}{\partial d_{ik}}$ is equal to zero at $\varepsilon = 0$ for any j, k so that $\frac{\partial H_i}{\partial d}(\varepsilon = 0) = 0$. In the next steps we will check that $det\left(\frac{\partial^2 H_i}{\partial d_i \partial \varepsilon}\right)_{|\varepsilon=0} \neq 0$ and require $\frac{\partial H_i}{\partial \varepsilon}(\varepsilon = 0) = 0$ to apply theorem 5: We can deduce from the above equations that $\frac{\partial^2 H_{ij}}{\partial d_{jk} \partial \varepsilon}(\varepsilon = 0) = u_i'' E[z_j z_k]$, i.e.

$$\frac{\partial^2 H_i}{\partial d \partial \varepsilon}(\varepsilon = 0) = u_i'' \cdot V.$$

(All other terms are equal to zero at $\varepsilon = 0$.) Therefore $det\left(\frac{\partial^2 H_i}{\partial d_i \partial \varepsilon}\right)(\varepsilon = 0) = u_i'' \cdot det(V)$ and $\left(\frac{\partial H_i^2}{\partial d \partial \varepsilon}(0)\right)^{-1} = \frac{1}{u_i''} V^{-1}$. Since $u_i'' > 0$ (u_i is strictly concave), and $det(V) \neq 0$ we have that $det\left(\frac{\partial^2 H_i}{\partial d_i \partial \varepsilon}\right)_{|\varepsilon=0} \neq 0$.

The condition $H_\varepsilon(\varepsilon=0)=0$ becomes the following system of equations:

$$\begin{aligned}0 &= \frac{\partial H_{ij}}{\partial \varepsilon}(0) = u_i'' \cdot \sum_{k=0}^{N} d_{ik}(0) Cov(z_j, z_k) + u_i' \cdot \pi_j(0),\\ \text{i.e. } 0 &= u_i'' \cdot V \cdot d_i + u_i' \cdot \pi(0) \text{ or equivalently } d_i(0) = \tau_i \cdot \left(V^{-1}\pi(0)\right).\end{aligned}$$

This gives us the zero order demand term. Now that the conditions have been checked to apply theorem 5 we calculate, since $E[\eta_i]=0$, that

$$\frac{\partial^2 H_{ij}}{\partial \varepsilon^2}(0) = u_i''' E[\eta_i^2 z_j] + 2u_i'' E[\eta_i]\pi_j(0) + 2u_i'\pi_j'(0) = u_i'''\zeta_j + 2u_i'\pi_j'(0).$$

Theorem 5 tells us

$$\frac{\partial d_i}{\partial \varepsilon}(0) = -\frac{1}{2}\left(\left(\frac{\partial^2 H_i}{\partial d_i \partial \varepsilon}(\varepsilon=0)\right)^{-1}\frac{\partial^2 H_i}{\partial \varepsilon^2}\right) = -\frac{u_i'''}{2u_i''}\left(V^{-1}\zeta_i\right) - \frac{u_i'}{u_i''}\left(V^{-1}\pi'(0)\right).$$

A.3 Calculating The Equilibrium

Using $d_i(0) = \tau_i \cdot (V^{-1}\pi)$ (equation 7) we get that the zero order aggregate demand is $(\tau_1+\tau_2)\cdot(V^{-1}\pi)$. This translates into (for securities $j=1,\ldots,N$)

$$\pi_0(0) = \frac{1}{\tau_1+\tau_2}V_{00}, \pi_j(0) = \frac{1}{\tau_1+\tau_2}V_{0j}, d_{i0}(0) = \frac{\tau_i}{\tau_1+\tau_2}, d_{ij}(0)=0.$$

This implies that $\eta_i = d_{i0}(0)z_0$ and so $\zeta_{ij} = E\left[\left(\sum_{k=0}^{N} d_{ik}(0)z_k\right)^2 z_j\right] = d_{i0}^2(0)E[z_0^2 z_j] = d_{i0}^2(0)\cdot\chi_j$. Therefore, the ζ_i vector reduces to the vector $d_{i0}^2(0)\cdot\chi$. We calculate the first order equilibrium demand and price terms using equation (7) as

$$d_i'(0) = V^{-1}\left(\tau_i\pi'(0) + \frac{\rho_i}{\tau_i}\left(\frac{\tau_i}{\tau_1+\tau_2}\right)^2\chi\right).$$

The market clearing condition for the first order demand is then:

$$0 = d_1'(0) + d_2'(0) = V^{-1}\left((\tau_1+\tau_2)\,\pi'(0) + \left(\frac{\tau_1\rho_1}{(\tau_1+\tau_2)^2} + \frac{\tau_2\rho_2}{(\tau_1+\tau_2)^2}\right)\chi\right)$$

which implies

$$\pi'(0) = -\frac{\rho_1\tau_1+\rho_2\tau_2}{(\tau_1+\tau_2)^3}\chi,$$

and so

$$d_1'(0) = \left(\frac{\rho_1\tau_1}{(\tau_1+\tau_2)^2} - \frac{\tau_1}{(\tau_1+\tau_2)^3}(\rho\tau_1+\rho_2\tau_2)\right)V^{-1}\chi = -\tau_1\tau_2\frac{\rho_1-\rho_2}{(\tau_1+\tau_2)^3}V^{-1}\chi.$$

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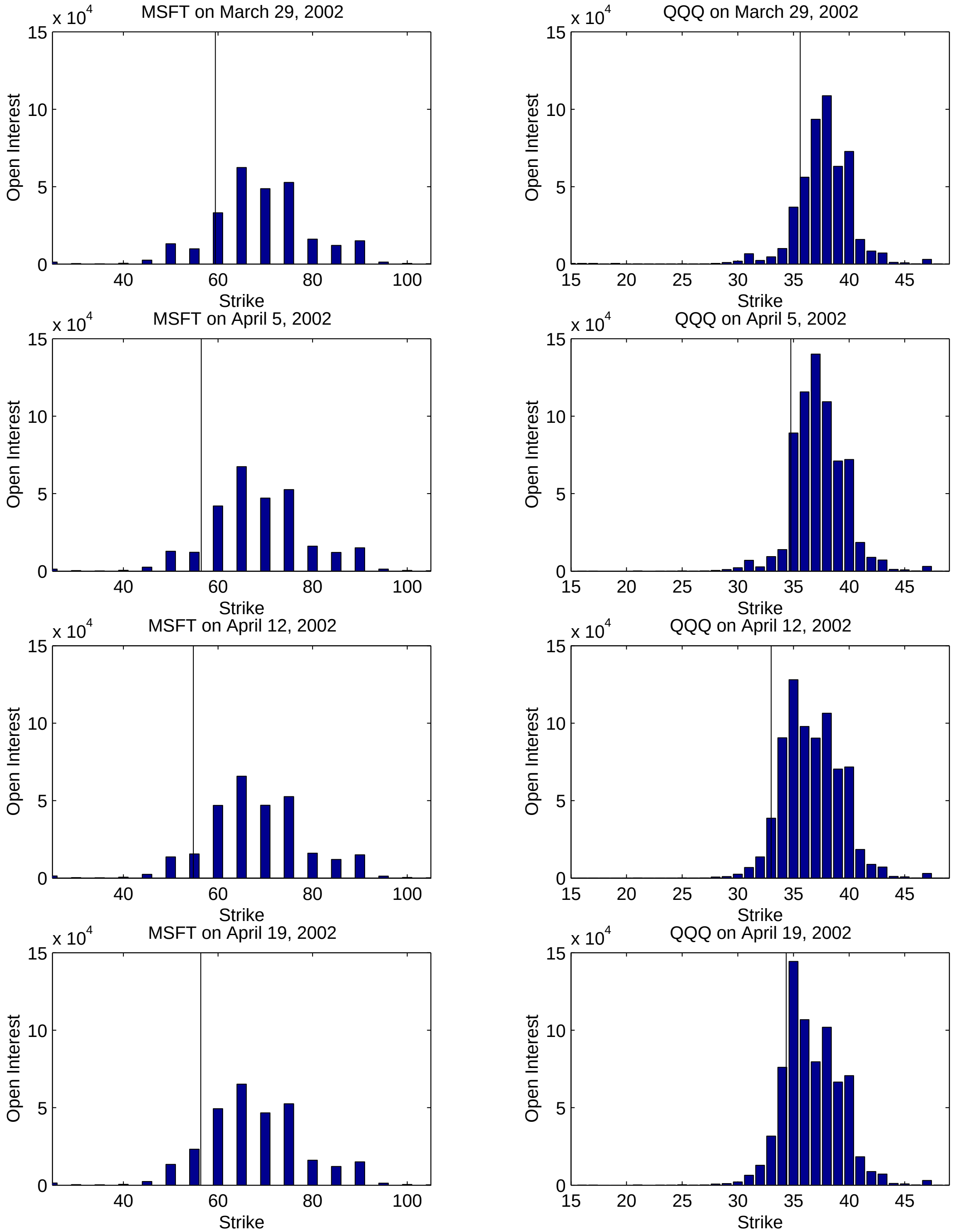

Figure 1: Empirically observed call option open interest in the April 2002 series for Microsoft (MSFT) and the NASDAQ 100 tracking unit (QQQ) on the last four Friday's before maturity. (The vertical line indicates the price of the underlying on that day.)

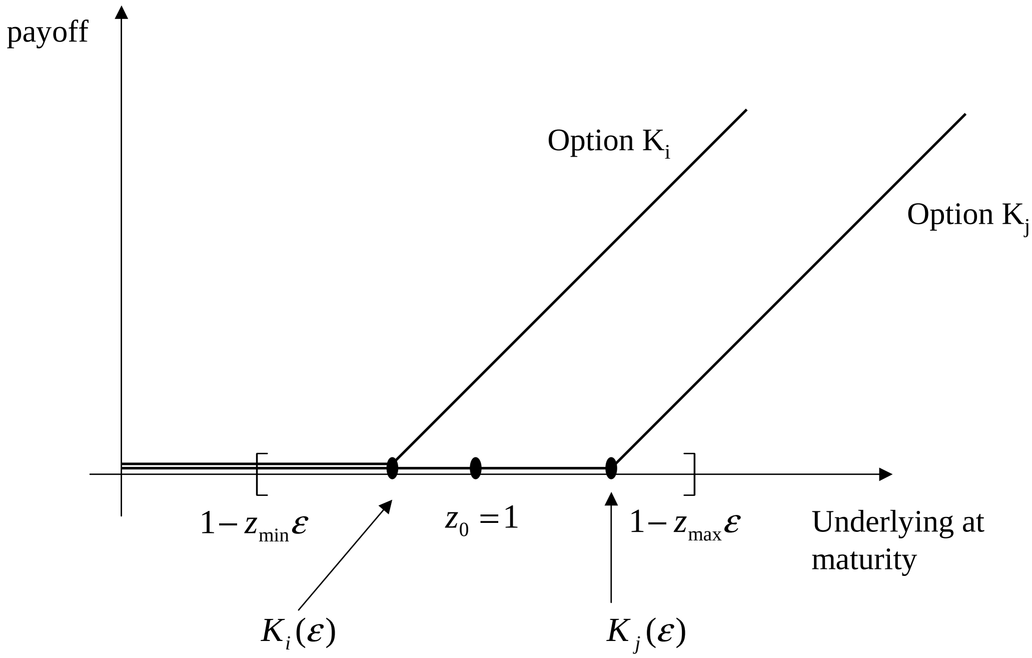

Figure 2: Payoff from options K_i and K_j in the ε-economy is that of a standard call with strike $1 + \varepsilon(K_i - 1)$ and $1 + \varepsilon(K_j - 1)$.

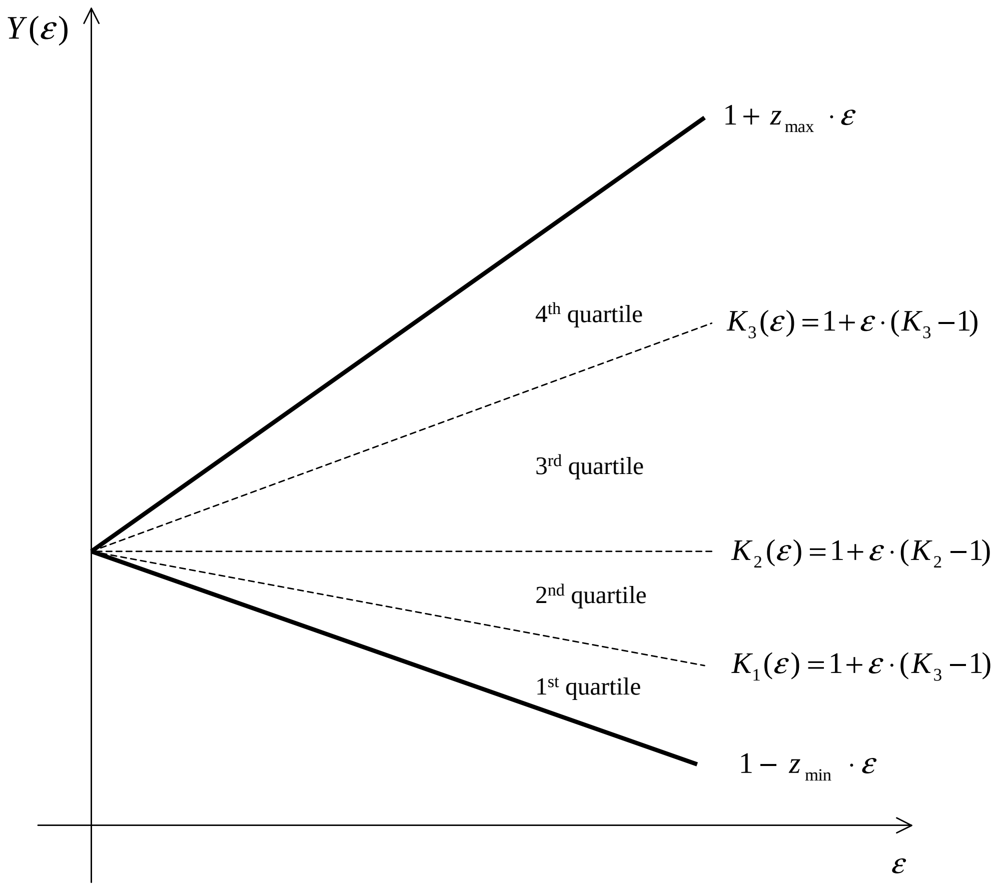

Figure 3: Expanding the strikes such that the probability of exercise remains unaffected: distribution with support bounded by z_{min}, z_{max} and strikes at the first three quartiles in the ε-economy.

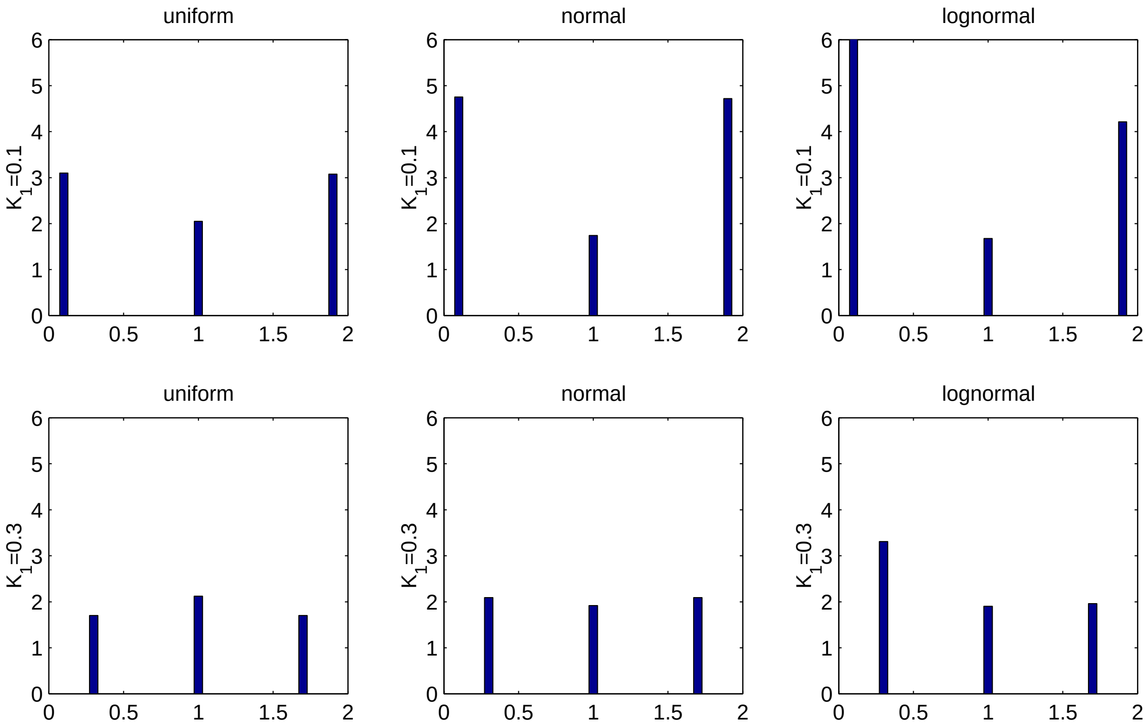

Figure 4: The shape is sensitive to the underlying distribution and the minimal strike parameter when three options are traded.

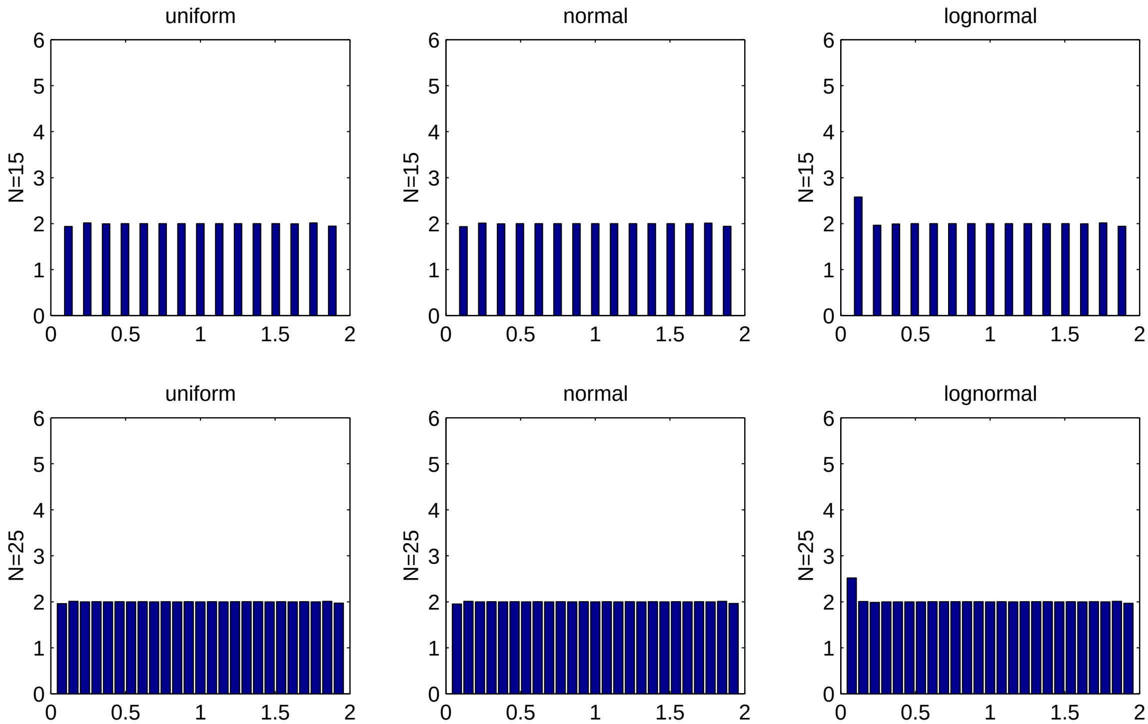

Figure 5: Increasing the number of options the open interest curve becomes flat.

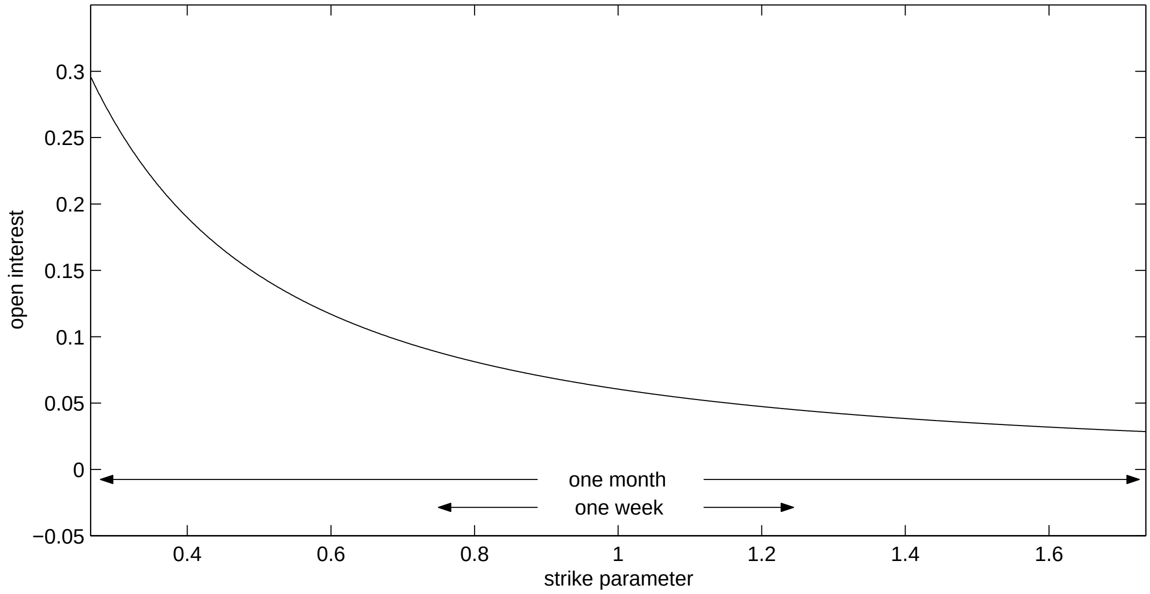

Figure 6: The open interest curve when the two agents have logarithmic and square root utility respectively.

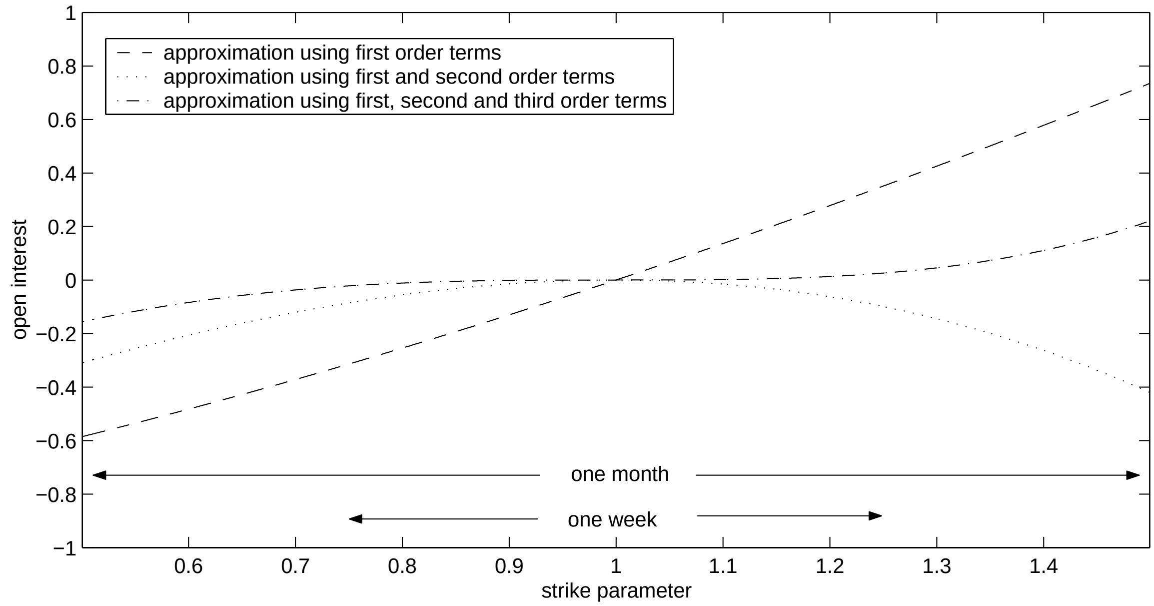

Figure 7: Relative errors in our approximation.

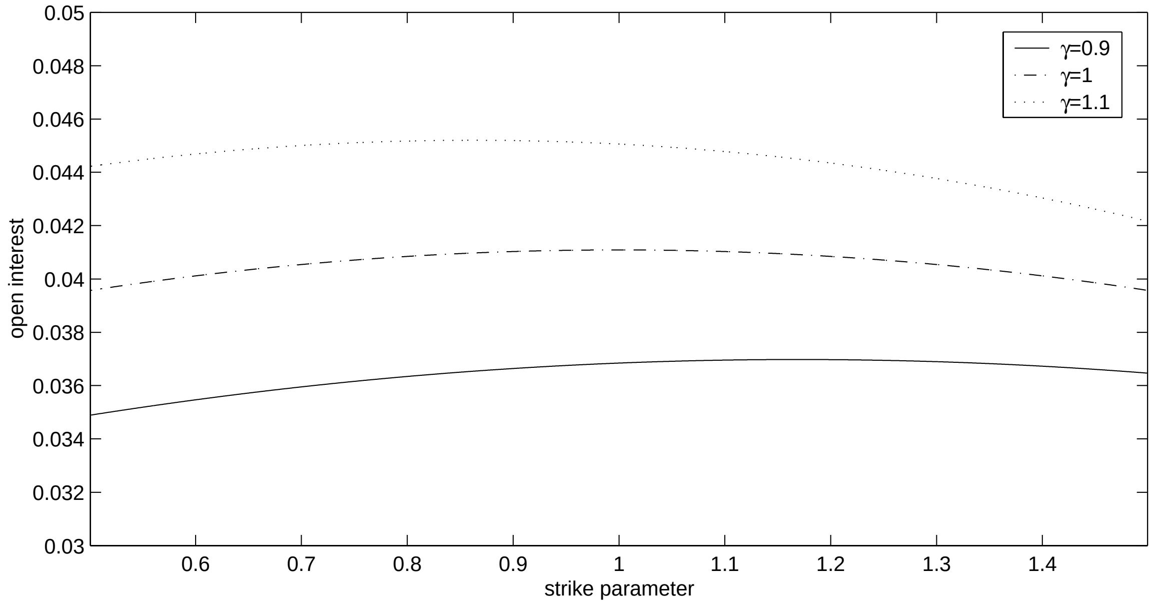

Figure 8: The open interest curve when the first agent has logarithmic utility and the second has exponential utility with parameter γ.