ESSAYS ON DYNAMIC ALLOCATION AND PRICING

A DISSERTATION
SUBMITTED TO THE DEPARTMENT OF ECONOMICS
AND THE COMMITTEE ON GRADUATE STUDIES
OF STANFORD UNIVERSITY
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

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August 1991

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Abstract

This dissertation is comprised of three distinct papers. The first, entitled "Time-Varying Risk Premium in the Foreign Exchange Market: Assessing Specification Tests and Measuring Model-Noise Error", is an examination of the efficiency of the foreign exchange market using a signal extraction approach. The inability of forward exchange rates to accurately predict future spot rates has continued to be one of the most puzzling features of the foreign exchange market. Many economists have tested the idea of "simple efficiency" in the foreign exchange market which requires that the investors have unbiased rational expectations and that the risk premium be non-existent. Rejection of these tests has therefore sometimes led researchers to believe that there exists an exchange rate risk premium. This paper reinvestigates the issue of efficiency in the foreign exchange market by assessing and ranking the relative power of the various specification tests. Different information sets used in the orthogonality tests encompass different implications of the null. Using a signal extraction framework, it is possible to interpret the various regressions in this context. The signal extraction framework shifts the emphasis away from merely testing for specification error, and thus rejecting the null, to measuring the deviation from the null model. It appears worthwhile to move away from being concerned only with the rejection of the null model to being equally interested in finding out the magnitude of the specification error that resulted in the rejection.

The second paper, "Illegal Immigration Under Heterogeneous Labor and Asymmetric Information", deals with the push and pull factors of illegal immigration. Given the apparent inability of policy-makers and researchers to deal effectively with the problem of illegal immigration, there is a need in the present literature to provide a theory of this labor flow. We construct a theoretical model of illegal immigration incorporating the two features of heterogenous labor and asymmetric information. It will be shown that when workers are heterogeneous, it is perfectly reasonable for source country employment changes to have a greater effect on the migration flow than host country employment changes. While changes in the demand conditions in the host country only strengthen the pull effect of the migration flow, source country

employment changes have both push and pull impacts on the rate of migration.

The third paper in this dissertation is a joint work with Kenneth Judd of the Hoover Institution and focuses on the existence of finite sample bias in generalized method of moments estimation The estimation of nonlinear rational expectations models has traditionally been plagued by difficulties. Primary among these has been the inability of researchers to obtain closed form solutions, in terms of the structural parameters, for the equilibrium time paths of the variables of interest. To circumvent this problem, Hansen and Singleton (1982) introduced an estimation procedure known as generalized method of moments (GMM) which exploits the population orthogonality conditions by choosing estimates of the parameters that lead the sample versions of these population conditions to be as close to zero as possible. Typically, hypothesis testing of the estimates derived from generalized method of moments rests on the asymptotic properties of these estimates. The goal of this paper is to demonstrate that small sample biases are in fact substantial. We illustrate this point in the context of a discrete time stochastic growth model that, unlike previous work in the literature, has the attractive feature of endogenous consumption and wealth Using Monte Carlo studies, we ascertain the magnitude of the bias and determine its relation to the different instruments and lag lengths in the information set

Preface

Along with the completion of this dissertation comes the pleasure of acknowledging a special group of people whose support over the past few years has been invaluable. My dissertation committee of Steven Durlauf, Robert Hall and Robert Staiger provided me with guidance and insight into my work. I cannot thank them enough for their countless hours of discussions and, in particular, for Steven Durlauf's continual encouragement, over the years . I owe a special thanks to Kenneth Judd for not only imparting to me his patience and knowledge but also for sharing with me his tremendous enthusiasm for economics and science.

My fellow students at Stanford have made graduate school a memorable experience. My deepest appreciation goes to Andrew Bernard for his unwavering friendship, his constancy, and his ability to make me smile and laugh. Special thanks must also go to Elisabeth Browne for her honesty, her friendship, and her patience. In addition, Paul Johnson provided not only helpful comments on my work but shared with me his sense of humour. Jessica Primoff inspired me with her diligence, while Kwang-Soo Cheong made those long hours in the office pleasant and enjoyable.

This dissertation is dedicated to my mother. She taught me to dream, and her love and support along with that of my brother, Alexander, gave me the encouragement to make this dream come true.

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Chapter IV. Finite Sample Bias of Generalized Method of Moments Estimation

1. Introduction

Although extensive work has been done with the specific case of dynamic rational expectations models in which agents are assumed to solve quadratic optimization problems under linear constraints, the estimation of nonlinear rational expectations models has traditionally been plagued by difficulties.[1] Primary among these has been the inability of researchers to obtain closed form solutions, in terms of the structural parameters, for the equilibrium time paths of the variables of interest. To circumvent this problem, Hansen and Singleton (1982) introduced an estimation procedure known as generalized method of moments (GMM). Standard method of moments estimation involves choosing admissible parameter values that minimize a weighted average of a number of the sample counterparts of the population conditions. A typical example is the case of a normal distribution with mean μ and variance σ^2 where, in the just-identified case, we might choose the sample mean and variance, respectively, as our estimates of these parameters. GMM techniques, which have become standard estimation techniques for testing nonlinear rational expectations asset pricing models, is a specific case of method of moments estimation. It exploits the population *orthogonality* conditions by choosing estimates of the parameters that lead the sample versions of these population conditions to be as close to zero as possible. When applied to the estimation of the stochastic Euler equations, it allows one to estimate parameters of the model and to test overidentifying restrictions of the model without solving for the decision rules. It is important to note, however, that one must still make assumptions on the functional form specifications surrounding tastes and technology in the economic environment.

Hansen and Sargent (1982) analyze linear rational expectations models in which agents forecast infinite geometrically-declining sums of the forcing variables.

Hypothesis testing of the estimates derived from GMM procedures typically, as discussed by Hansen and Singleton, rests on the asymptotic properties of these estimators. When one is dealing with macroeconomic variables, however, one may be skeptical of making inferences based on asymptotic distribution theory. Other works which have used alternative approaches to the numerical solution of these nonlinear models have found the existence of finite sample bias in GMM estimation procedures. The objective of this paper is to go further in demonstrating that small sample biases of GMM estimation are indeed substantial. We illustrate this point in the context of a discrete time stochastic growth model that, unlike previous work in the literature, has the attractive feature of endogenous consumption and wealth. Using Monte Carlo studies, we attempt to ascertain the magnitude of the bias and to determine its relation to the different instruments and lag lengths in the information set. In addition, we investigate the correlation, if any, between the magnitude of the bias and the degree of risk aversion in the utility function.

Tauchen (1986) and Kocherlakota (1989) both examine the finite sample properties of these method of moments techniques in nonlinear asset pricing models. Tauchen examines the finite sample properties of GMM estimators using a simple asset pricing model with one asset. The main idea behind this work is to build a small-scale artificial economy patterned on the asset pricing model in Lucas (1978). Lucas models a one-good, pure exchange economy with identical consumers. The good is produced costlessly in different productive units where the productivity in each unit fluctuates stochastically through time. Output is perishable and therefore consumption at time t must be less than or equal to output at time t. A single representative consumer maximizes $E\{\sum \beta^t U(C_t)\}$ where E is the expectation operator, C_t is consumption governed by an exogenous stochastic process, β is the discount factor and $U(\cdot)$ is the current period utility. Tauchen uses this one-agent economy with externally given stochastic laws of motion for consumption and asset dividends. One difference between the model in Tauchen and that in Lucas is that Tauchen defines consumption and the M-asset dividends as having a non-degenerate $(M+1)$ dimensional probability distribution while in Lucas, the consumption endowment is defined as the sum of the dividends implying a singular joint distribution for dividends and consumption. However, one can imagine that there exists an $M+1$ asset and that consumption is the sum of all the $M+1$ dividends; in which case, there is no difference between the two set-ups. In particular, consumption follows a process calibrated to be similar to U.S. data. Using numerical procedures for solving integral equations, Tauchen computes the equilibrium solution for asset prices as a function of current and lagged income. Given the assumed consumption process, the computed solution for asset prices, and random realizations of exogenous shocks, he computes a simulated time series of consumption and asset prices He then applies standard GMM procedures to the simulated data. He generates several such simulated data sets and estimations, thereby building up a sample distribution of parameter estimates and standard errors. With this empirical distribution in hand, he compares the known true parameter values to their estimates, and the standard deviation of the estimates to the asymptotic standard error computed by GMM procedures.

Tauchen assumes a CRRA utility function in his paper and chooses the constant relative risk aversion parameter γ to be -0.30 and -1.30. He finds that under certain circumstances there is bias in the estimate of γ with the bias being at least as large as the asymptotic standard error The direction of the bias appears to depend on the covariance structure of the dividend growth process and the consumption growth process. Widely divergent parameter estimators result when different lag lengths are imposed on the instrument set. Short lags used in forming the instrument sets result in nearly asymptotically optimal estimates. However, as the lag lengths increase, the sampling distributions of the estimates become more and more concentrated around severely biased values The sampling distribution of the estimators shows a variance/bias tradeoff as the number of lags used in the instrument set increases. As the number of lags rises, sample estimates are less dispersed but the bias rises. Therefore, Tauchen suggests that among a large set of estimates produced with different information sets, the most reliable estimates are those obtained with the *smallest* instrument set, i.e., shortest lag length, because the confidence intervals of those estimates will be more reliable. This is especially true for any loss function that penalizes heavily for incorrect inferences that arise from a bias that is large relative to the standard error. Tauchen finds that for short lag lengths, the confidence intervals are very reliable as the coverage rate is close to the anticipated rate of 0.95. Furthermore, tests of overidentifying restrictions perform well in moderately-sized samples.

Kocherlakota also tests for the finite sample bias of GMM procedures using the basic methods of Tauchen. He examines a single agent economy where the process governing the growth rates of consumption and dividends is an N-state Markov chain. The specification of the stochastic process of stock market dividends and aggregate consumption is found by fitting a vector autoregression to accord with annual data for the United States economy. Unlike Tauchen's model, Kocherlakota includes a riskfree asset in his simulations. The model is thus calibrated so as to mimic real data. A coefficient of constant relative risk aversion of -13.7 is chosen This large degree of risk aversion is consistent with the findings by Hansen and Jagannathan (1989) who note that in a correct model of asset pricing, the standard deviation of the intertem ral marginal rate of substitution of the representative agent should be large relative to its mean Kocherlakota shows that for GMM estimators typically used, the small sample distribution of the J-statistic stochastically dominates its large sample distribution, and therefore it tends to lead to overrejection of the model

For each estimator, Kocherlakota generates 400 data sets of 90 observations each on the growth of consumption and the asset returns using the probability structure of his model economy. The numerical procedure always begins at the true parameter values. In this case, the preference parameters consisting of the discount factor and the coefficient of constant relative risk aversion are set at 1.139 and -13.7 respectively. The tests are performed over 7 different sets of instruments, resulting in seven different estimators. Estimators 1 through 3 are based on actual annual data and use multiple instruments. By multiple instruments, he refers to returns on the stock market, the riskfree rate of return and the growth rate of consumption. Kocherlakota finds that these estimators with multiple instruments perform poorly in the small sample data set They tend to bias the preference parameters towards zero. Furthermore, the sample distribution of the J-statistic is not approximately chi-square. Assuming that it is leads to overrejection of the model at the 5 percent significance level. In other words, suppose that one assumes that the J-statistic has a chi-square distribution, a much lower significance level of test should be used in order to have a 5 percent rejection rate. However, Kocherlakota also finds that 2 of the 3 estimators with multiple instruments do converge slowly to their large sample properties. At the same time, Kocherlakota notes that the discount factor greater than 1 cannot account for the poor small sample properties since, in general, lowering the discount rate and raising the CRRA worsen the estimates. Estimators 4 through 7 which involve only one instrument are shown to perform well. These estimators are based on an instrument set that contains one of any of the three instruments: the stock market return, the riskfree rate of return or the growth rate of consumption

In order to test for finite sample bias, we must be able to first solve and then estimate these nonlinear models. As alluded to earlier, the analysis performed in this paper goes a step further than Tauchen and Kocherlakota in that we allow consumption, wealth and income to be endogenous. In our model, wealth, consumption and asset returns are not deterministically related as they are in the analyses performed by Tauchen and Kocherlakota. Tauchen assumes that the law of motion for income is exogenously given and that the consumption process follows that of income. He then uses the first order condition to solve for the implied asset prices. Our model is much more realistic in allowing consumption and the capital stock to be endogenously determined. Consumption is completely described by the two state variables, the capital stock and the productivity shock

Briefly, the methodology is as follows. We stipulate a nonlinear discrete time model of stochastic growth and calculate the decision rules for consumption By solving this model, we can simulate time series of consumption, production, capital stock and asset returns. We then perform generalized method of moments estimation on these simulated data sets and test whether the estimated parameters are centered around their true values. We provide statistics on the sample distribution of the estimators as well as on the J-statistic testing the overidentifying restrictions of the model

One of our main objectives is to determine how sample bias depends on the instrument set chosen, or whether sample bias is an inherent result of the estimation procedure and independent of the choice of instruments. Typically, we project the Euler equation error against lagged consumption and lagged asset returns. Estimators that employ several instruments are more common than estimators which only use one single instrument although Kocherlakota appears to find that a single instrument leads to better parameter estimates than using multiple instruments. However, the same orthogonality arguments should apply to *nonlinear* functions of the lagged consumption and lagged asset returns data. In fact, there is no theoretical basis to restrict the information set to linear functions of the elements. One way of adding in these nonlinear functions is through the incorporation of higher powers of lagged consumption and lagged asset returns.

Furthermore, rather than use only consumption data and asset returns in the instrument set, in this model it may be more appropriate to include capital stocks as well. This is because the capital stock equals the total value of the assets. The dividend variable that is typically included in generalized method of moments estimations equals, in our model, the capital stock multiplied by the asset returns.

Therefore, in addition to simulating a model with endogenous consumption, the question we pose is whether by adding new orthogonality conditions, generalized method of moments would be better able to detect deviations from the Euler conditions and whether finite sample bias would be reduced. We will also attempt to determine whether with these added instruments, the results obtained by Tauchen concerning the optimal lag lengths of the instruments continue to hold true

Section 2 describes the growth model and the assumptions underlying this model economy. Section 3 outlines the numerical algorithm we use to solve this discrete time stochastic growth model. In section 4, we briefly review the GMM methodology and then apply it to the data generated in this model economy. Section 5 provides our results. We conclude in Section 6.

2. Discrete Time Stochastic Growth Model

We will examine the Brock-Mirman discrete time model of stochastic growth under an AR(1) productivity specification. Agents maximize the present value of expected lifetime utility:

$$E\left\{\sum_{t=0}^{\infty}\beta^t U(C_t)\right\}$$

where

$$\begin{aligned} U(C_t) &= \frac{C_t^{1+\gamma}}{1+\gamma} \quad \gamma<0, \quad \gamma\neq -1 \\ &= \ln \quad C_t \quad \gamma=-1 \end{aligned}$$

where C_t is consumption at time t and γ is the coefficient of constant relative risk aversion. The underlying economy is described by the following equations:

$$K_{t+1} = \theta_t f(K_t) - C_t$$

$$f(K_t) = aK_t^{\alpha}$$

$$\ln\theta_{t+1} = \rho\ln\theta_t + \epsilon_{t+1}$$

where K_t is the capital stock at time t. θ_t is defined as a stationary AR(1) multiplicative productivity parameter and $\epsilon_t \sim N(0,\sigma^2)$. Both the initial capital stock K_0 and the initial value of θ, θ_0, are exogenous.

One property of this model is that the capital stocks and productivity shocks completely describe the state of the economy. One can deduce the capital stock for the $t+1$ period from the capital stock of the previous period t and the productivity shock of the present period $t+1$. Production and current consumption can then easily be derived from the capital stock variable. This implies that consumption decisions are a function, $h(K,\theta)$, of current values of K and θ. The decision rule must satisfy the Euler equation

$$U'(h(K,\theta)) = \beta E\left\{U'(h(\theta f(K) - h(K,\theta),\bar{\theta}))\bar{\theta} f_k(\theta f(K) - h(K,\theta)) \mid \theta\right\} \quad (2.1)$$

where the decision rule is denoted by $h(K,\theta)$ and $\tilde{\theta}$ is the productivity shock one period forward. Since we know that this algorithm that we will be using to solve the model works well for linear problem, we rewrite (2.1) as

$$0 = h(K,\theta) - (U')^{-1}\left(\beta E\left\{ \qquad \right\}\right) \tag{2.2}$$

so that it looks more like a linear problem. The RHS now has two terms, one linear and the other similar to a constant returns to scale function of next period's potential consumption values.

3. Minimum Weighted Residual

Our analysis can be broken down into two stages. In the first stage, we solve for the decision rules for consumption In the second stage, after generating the simulated data series using these decision rules, we estimate the values of the parameters of interest through GMM procedures. The class of techniques which we will use in this paper to solve this growth model is known as Minimum Weighted Residual (MWR) and is carefully explained in Judd (1991). This method was developed in the mathematical literature to solve for numerical solutions to partial differential equations. Its wide application to economic problems has been demonstrated in Judd (1991). The idea is to express the equilibrium as a solution to some operator equation, such as an Euler equation or a differential equation. We then specify a topological space[2] which contains the solution and look for an approximate solution in a finite dimensional subspace of that space. This algorithm is especially appropriate for our purposes in that it is a fast algorithm. Since empirical analysis requires repeated use of the algorithm, a fast algorithm allows us to feasibly perform econometric estimation of the underlying model. The algorithm is performed in FORTRAN using LINPACK's

[2] Imposing the assumption of a topological space is unnecessary for the purposes of this paper but will be relevant should one attempt to prove convergence of our polynomial estimates.

SNSQE on a VAX.[3] We present here a succinct description of the Minimum Weighted Residual algorithm specifically in the context of the discrete time stochastic growth model we are fitting.[4]

We first define the problem which is represented as a solution to the nonlinear operator equation

$$\mathcal{N}(f) = 0$$

where $\mathcal{N}$: $B \to B$, B is a Banach space of functions $f : D \subset R^n \to R^l$. The domain D will represent the N state variables, the unknown function f will represent the decision rules and $\mathcal{N}$ will represent the M Euler conditions. Recall that in our model, consumption defined as $h(K,\theta)$ is the unknown function and it must satisfy equation (2.2):

$$0 = h(K,\theta) - (U')^{-} \left(\quad \left\{ U'(h(\theta f(K) - h(K,\theta), \tilde{\theta}))\tilde{\theta} f_k(\theta f(K) - h(K,\theta)) \mid \theta \right\} \right)$$

which is the error to the Euler equation.

We now show how to implement MWR in a step-by-step manner.

S1: Choose a set of parameters Γ to estimate:

$$\Gamma = \{\beta, \gamma\}$$

In this model, our parameters of interest are the discount factor β and the coefficient of constant relative risk aversion γ. We may, if we choose to, also include in this set α, the capital share, σ^2, the variance in the productivity shock, and ρ, the serial correlation of the productivity shocks.

S2: Assign initial values to Γ and to any other taste and technology parameters.

In our estimations using monthly series, we set

$$\sigma = .008$$

[3] See Judd (1991) for more details on the speed and accuracy of this algorithm.

[4] Other numerical techniques which have been used in the literature to deal with non-linear systems that tend to arise in dynamic economic models with rational agents are discussed in Judd (1991).

$$\alpha = .3333$$

$$\beta = .9957$$

$$\gamma = -.5, -2.0, .., -10.0$$

Since we are simulating monthly data, a β of 0.9957 corresponds roughly to an annual discount factor of 0.95. We solve for the decision rules of consumption for a range of γ' s.

S3: Express the equilibrium as a solution to some operator equation.

In our example, equilibrium is expressed as the solution to the Euler equation (2.2) above.

S4: Specify the topological space which contains the solution to the model and derive the decision rules $h(K, \theta)$. Specifically, this involves the following steps:

S4.1: Choose a basis, $\Phi = \{\varphi_i\}_{i=1}^{\infty}$, of the space of continuous functions where we will search for the approximate solution and the norm.

The basis should be flexible, capable of yielding a good approximation to the solution, and the inner product $< \cdot, \cdot >$ should induce a useful norm on the space spanned by Φ. The considerations that should go into choosing a basis are discussed in Judd (1991). Briefly, a basis should be easy to compute and be orthogonal relative to a relevant norm. Furthermore, the basis elements should resemble the solution so that only a few elements will be sufficient to generate a good approximation. We could conceivably have used ordinary polynomials $\{1, x, x^2, x^3 ...\}$ but they are all monotonically increasing and positive on R^2. They will not be orthogonal in any natural norm since they are potentially very similar. The reason we prefer terms that are orthogonal with respect to the inner product $< \cdot, \cdot >$ is essentially the same as why one wants uncorrelated explanatory variables in a regression. Nonorthogonal bases reduce numerical accuracy just as multicollinear regressors enlarge confidence intervals. The bases we therefore select here are the Chebyshev polynomials. The Chebyshev polynomials are defined over [-1,1]. A Chebyshev polynomial of degree n is denoted by $T_n(x)$ and is given by the formula

$$T_n(x) \equiv \cos(n \arccos x)$$

They are generated by the recursive scheme

$$T_0(x) \equiv 1$$

$$T_1(x) \equiv x$$

$$T_2(x) \equiv 2x^2 - 1$$

$$T_{n+1}(x) \equiv 2xT_n(x) - T_{n-1}(x) \quad n \geq 1$$

The Chebyshev polynomials form a suitable basis set in that they obey the discrete orthogonality relationship

$$\sum_{k=1}^{m} T_i(x_k)T_j(x_k) = 0, \quad i \neq j$$

where

$$x_k \equiv cos\left(\frac{\pi(2k-1)}{2m}\right), \quad k = 1, ...n$$

It is not the case that Chebyshev polynomials are necessarily more accurate than some other approximation polynomial of the same order n but that the Chebyshev polynomial can be truncated to a polynomial of lower degree m in such a way as to yield the most accurate approximation of degree m.

We allow for two state variables, K, the capital stock and θ, the productivity shock. We therefore need a basis for a function of two variables. This basis function is built from the one-dimensional case by constructing a tensor product of the one dimensional basis.

$$\{\varphi_i(K)\varphi_j(\theta)\}_{i,j=1}^{\infty}$$

For problems with n-dimensions, one can take the n-fold tensor product of a one-dimensional basis. One advantage of the tensor product approach is that if the one-dimensional basis is orthogonal, so is the tensor product. The disadvantage is that as the dimension increases, the number of elements increases exponentially.

Choose a degree of approximation n_i for each state variable i.

We have chosen an n_k of 7 for the capital state variable and an n_θ of 5 for the ductivity shock variable. The only correct value for n is ∞ but for computation urposes, it is in our interest to choice the smallest n_i to give us a good approximation. ı previous work by Judd (1991), he shows that approximations of low order appear › be sufficiently accurate. We choose for this model to let $h(K,\theta)$ be a *sixth*-order olynomial in K and a *fourth*-order polynomial in θ.

S4.3: For a guess $\vec{a}$, compute the approximation, $\hat{f} \equiv \sum_{i=1}^{n} a_i \psi_i(K,\theta)$, and the residual

$$R(K,\theta;\vec{a}) \equiv \mathcal{N}(\hat{f})$$

We specify the residual set of vectors R as the errors to the Euler equations. We attempt to find an $\hat{f}$ that fits (2.1) with the smallest error. The first guess of $\vec{a}$ should therefore reflect some knowledge about the solution.

S4.4: Choose and compute the projection conditions used to identify $\vec{a}$.

There are a variety of ways to choose the projection conditions. They include the method of moments, subdomain and collocation, all discussed in Judd (1991). Projections generally involve integration which is often difficult to do for nonlinear economic problems. Therefore, a feasible alternative is to use quadrature formulas which are essentially weighted collocation methods since they involve evaluating the integrand at a finite number of points. The advantage of quadrature formulas is that information at more points is used to compute the approximation of the projections.

S4.5: Iterate over steps 4.3 and 4.4, to find the $\{a_i\}_{i=1}^{n}$ that sets the projection equations to zero.

Once we have solved for the $\{\hat{a}_i\}_{i=1}^{n}$, we can solve for consumption at each state of capital stock and productivity shock.

Once we have numerically solved this model, we move onto the simulation and estimation stages of this analysis. With the computed decision rules, we can create many *simulated* data sets to test for finite sample bias of our estimators. Recall that our model does not include labor supply, instead treating that as fixed or exogenous, although it can be incorporated in this model. We, therefore, chose not to test the simulated data set here against actual realized consumption and production data. Thus we first simulated a reference data set.

In the Monte Carlo study, we run simulations on a number of different sample lengths. *Nruns* number of runs were performed for series of lengths 240 and 24000 respectively. These lengths refer to the number of months in the sample period and correspond to time series paths of 20 years and 2000 years. β is chosen so that the annual discount factor is 0.95. We had initially performed the exercises on series of length 1200 months, i.e., 100 years, but found that the results between these runs and those of length 240 months were very similar. We also performed runs on a set of annual data. In each case, we obtain a sample distribution of the parameter estimates and their standard errors. Our objective is to track the sample distributions of these estimates as we increase the sample size. We estimate the coefficient of the constant relative risk aversion parameter γ and the discount factor β using generalized method of moments. Unlike Tauchen, however, we include nonlinear functions of the instruments in the instrument set. One question we are attempting to address is whether the finite sample bias found by Tauchen is a feature of the estimation procedure itself or a result of the poor choice of instruments. If we find that the small sample bias has not improved, then we may conclude that nonlinear functions of the instruments set do not improve the efficiency of this estimator.

S5: Generate a simulated data set given the decision rules obtained in S4 and treat this as the *reference* data.

S6: Simulate *nruns* data series.

We now use the decision rules derived in the numerical section above to simulate the consumption, asset returns, capital stock, and production data series. Uncertainty is introduced through the productivity shock that we had defined earlier as following an AR(1) process:

$$\ln \theta_{t+1} = \rho \ln \theta_t + \epsilon_{t+1}$$

$$\rho = 0.9$$

To mulate $\boldsymbol{\theta}$, nitialize θ_0 id generate random series of dr from ormal distribi ti wi mean id vari σ^2 which again choice parameter. With his of $\boldsymbol{\theta}$, ula he capital stock and prod ction level. We discard the first 00 periods of mulated data to ensure that th itial val of will not affect res ilts. The consumpti tream constructed using the $\hat{a}_{i}$ solved for earlier the Chebyshe polynomials

S Choose set of moment condi ions for the simulated data set $M(\Gamma)$

For this paper the choose are th orthogonality projections. We perform these projections using differen strument sets The instrument sets include lagged consumption capital tock production id the marginal prod ctivity of capital Another choice for $M(\Gamma)$ may be combine orthogonali conditions with ice id covariances of the time es although unclear how should th calculate th timal ghting matrix.

4. Generalized Method of Moments

Here, we first provide a brief overview of the GMM technique and then proceed to explain any modifications made in this paper. Hansen and Singleton (1982) developed the GMM technique for estimating the underlying parameters of a representative agent's utility function. This involved the orthogonality conditions implied by the first order conditions for maximizing lifetime utility subject to the budget constraint. For any such intertemporal asset pricing model, the conditional expectation of the Euler equation error should be zero when evaluated at the true parameter value. Given data on consumption and asset returns, define Γ as the set of parameter values with Γ_0 being the set of true parameter values. Let e_t be the error to the Euler equation and z_t a set of instruments in the I_t information set of agents. By construction it must be true that the error at time t be uncorrelated with variables in the information set of time t. Letting E_t be the conditional expectation operator at time t

$$E_t\{e_t(\Gamma_0)|z_t\} = 0$$

Rewriting $e_t(\Gamma_0) \otimes z_t$ as $g_t(\Gamma_0)$,

$$E_t\{g_t(\Gamma_0)\} = 0$$

Define

$$\bar{g}_T(\Gamma) = (\frac{1}{T}\sum_{t=1}^{T} g_t(\Gamma))$$

as the sample average of the $g_t(\Gamma)$ where T is the sample size. Then as $T \to \infty$, $\bar{g}_T(\Gamma)$ must converge almost surely uniformly to $E\{g_t(\Gamma)\}$ under regularity conditions. In particular, GMM minimizes the quadratic form

$$Q_T(\Gamma) = \bar{g}_T(\Gamma)' W_T \bar{g}_T(\Gamma)$$

where W_T is a symmetric nonsingular weighting matrix that satisfies $W_T \to W$ almost surely where W is symmetric and nonsingular.

GMM parameter estimates are therefore complicated nonlinear functions of the data. The nonlinearities arise directly through those introduced in the objective functions and indirectly through the second step of the estimation procedure, the estimation of the optimal weighting matrix. Implicit in the technique, however, is the important assumption, noted by Garber and King (1983), that the functional form of the agent's objective function is known to the econometrician and not subject to shocks and fluctuations.

Since GMM is classified as an application of instrumental variable estimation, it is useful to review some more findings on the small sample properties of the IV estimator. Nelson and Startz (1990) addressed the issue of how finite sample bias varies with the quality of the instruments. Consider the regression $y = \beta x + u$ to be estimated with instrument z. A good instrument is one where the instrument is highly correlated with the regressor x and uncorrelated with the regression error u. If the correlation between z and x is small, the normal approximation is asymptotically valid, but a poor approximation to the true distribution in small sample. Their findings show that the finite sample distribution of the IV estimator is bimodal. Furthermore, with poor instruments and a small sample size, the asymptotic approximation is a poor approximation. The distribution of $\hat{\beta}$ may be quite concentrated around a point away from the true parameter value. In an accompanying paper, Nelson and Startz (1990) note that the conventional wisdom is that the consequence of having a poor instrument is a large standard error and a low t-ratio. However, in fact, the consequence may be more damaging in that the bias in the estimated coefficient will be large relative to its calculated standard error.

Given this brief review of GMM estimators, our next step is to apply the procedure to our simulated data series.

S8: Perform generalized method of moments estimation on the simulated data set to estimate the parameters of the model.

Traditionally, in asset pricing models of this type, the instrument set includes lagged consumption and lagged rates of return. We include in the information set nonlinear functions of these variables as well as those of lagged capital stock. The type of nonlinearity we have chosen to introduce is to take higher powers of these variables by decomposing them into orthogonal Chebyshev polynomials. It is onto these polynomials that we project the Euler equation error

$$< R, T(z) >$$

S9: Plot the estimates of β and γ for the different information sets and check for the existence of finite sample bias. Calculate the moments of the sample distribution of β and γ as well as the J-statistic on the overidentifying restrictions of the model.

Steps S1 through S9 provide a brief outline of the methodology we use to solve this discrete time stochastic growth model and to estimate the parameters of interest to us.

4. Results

We will discuss first our results from the simulation section and then the estimation stage of the analysis.

A. Simulations

From the stochastic simulations of consumption, capital stock and asset returns, we provide a number of descriptive statistics. In Table 1, we provide four summary statistics. They include the following:[5]

1. The statistic m suggested by Den Haan and Marcet (1989) provides a test for the martingale difference property $E_{t-1}\eta_t = 0$ which is satisfied by the theoretical solution. η_t is the residual of the Euler equation and the statistic m equals

$$\hat{a}(\sum x_t' x_t)(\sum x_t' x_t \eta_t^2)^{-1}(\sum x_t' x_t)\hat{a}$$

[5] All four summary statistics are suggested in Taylor and Uhlig (1990) as a means to differentiate among the many numerical techniques used to solve nonlinear rational expectations models.

where

$$\hat{a} = (\sum x_t' x_t)^{-1} (x_t' \eta_t)$$

is the ordinary least squares estimator in a regression of the Euler equation residual on a list of regressors which include a constant and five lags of consumption and θ. Using the asymptotic distribution, a two-sided test at a significance level of 2.5 percent for each side would be 3.82< m <21.92.

2. TR^2 from the regression of ϵ_t on five lags of consumption, capital and θ. This statistic tests for the martingale difference property $E_{t-1}\epsilon_t = 0$. Since TR^2 was constructed using 15 regressors plus a constant term, TR^2 has an asymptotic $\chi^2(15)$ distribution. The two-sided test at a significance level of 2.5 percent for each side is given by 6.26< TR^2 <27.49.

3. R^2 from the regression of the first difference of consumption on both lagged consumption and capital. This tests the random walk hypothesis of consumption. Results from Taylor and Uhlig (1990) note that among nearly all of the numerical solutions derived, the tabulated R^2 of the random walk declines as the coefficient of relative risk aversion rises.

4. Ratios of the variance of investment to the variance of the change in consumption. This ratio is a measure of the relative volatility of consumption and investment

In our 21 simulated cases, 6 cases satisfy the martingale difference property $E_{t-1}\eta_t = 0$ while 18 cases satisfy the martingale difference property for ϵ. The tabulated R^2 of the random walk remains fairly constant over all ranges of γ but does appear to decline as the coefficient of relative risk aversion moves from -0.5 to 2.0, thus being consistent with the results from Taylor and Uhlig (1990).

In Figure 1, we also illustrate the density function of the simulated consumption data series for length of 50000 months.

B. Estimation

In this section, we present our results on a sample set of simulations. Our first set of Monte Carlo studies is with an information set that consists of linear functions of the instruments, i.e., lagged consumption, lagged marginal productivity of capital and lagged capital stock. We run all our simulations over a γ range of -0.5 to -10.0. For a sample period of 240 months, we choose β=0.9957 and σ=0.008. Our findings on the point estimates show that, in the case of one lag, the estimated β coefficient is unbiased throughout while the γ coefficient exhibits a high degree of bias. While true values of γ range over [-0.5,-10.0], the estimated γ's fall predominantly between 1.0 and -2.0. As the true value of γ approaches -10.0, the number of true γ's that fall within the 95 percent confidence interval fall. Furthermore, the standard errors increase as the true value of γ approaches -10.0. In a graph with the true γ's on the horizontal axis and the estimated γ's on the vertical axis, the resulting plot is a horizontal line centered around 0.2, as shown in Figure 2.

When we replace the linear instrument set with nonlinear functions of the instruments, i.e., increase the order of polynomial to 2, the results do not appear to improve. In fact, the estimators appear to perform even worse. The estimated value of γ continues to center around zero while the true values of γ fall increasingly outside the 95 percent confidence interval.

We next perform a Monte Carlo study on a sample period of 40 years. The parameters chosen are β=0.95, ρ=0.5 and σ=0.02. As we can tell form Figure 4 and Figure 5, the results are very similar to those obtained from the monthly data series. However, when we increase the sample size to length 240 years, we obtain strikingly different results. With a polynomial order of 2, while in the case of monthly data, the estimated γ's centered closer to 0, they seem to fall close to -2.0 with annual data. The percentage of estimated γ's that fall within the confidence intervals also increases. The degree of bias is, however, still severe for lower values of the coefficient of relative risk aversion. Our brief sampling of lag lengths greater than 1 appear to support Tauchen's findings that longer lag lengths tend to worsen the estimates. Furthermore, we also find that with longer lag lengths, the sampling distribution ecomes more and more concentrated around severely biased values of γ.

Given that the GMM technique appears to have a severe finite sample bias, we erform the same Monte Carlo exercise with a longer sample of 24000 months. Our ojective here is to get a general idea of the range of sample length over which the asymptotics theories will hold, and to check whether the asymptotic properties hold for all values of γ. Our results are somewhat startling. With such a long sample of 24000 months, the estimator performs relatively well for high and moderate values of the coefficient of relative risk aversion. The bias is small and very close to the true value for $\gamma > -5.0$. With γ values smaller than -8.0, the bias rises as the estimated γ's appear to level off between -6.0 and -8.0. However, over the entire range of true γ values, the estimated γ's all fall within the 95 percent confidence intervals. It is somewhat surprising that with as long a sample as 24000 months, our estimators are still biased. It is difficult to believe that with macroeconomic variables, 2000 years is too short a time period to capture asymptotics. We have not performed the Monte Carlo simulations with any sample greater than 2000 years. However, with our studies of 240, 2400 and 24000 months, it is clear that although 24000 months is not a long enough period to approximate asymptotics, asymptotics do begin to hold as we increase our sample length.

With the long sample of 24000 months, the J-statistics calculated show that in 21 cases of different γ's, the overidentifying restrictions were rejected in 5 cases at the 5 percent significance level. With a lag length of 1 and an order of polynomial of 1 in the instruments, the number of degrees of freedom for the χ^2 statistic is 2. If a critical region of size 0.05 is chosen, in those 5 cases, the χ^2 value exceeds the value 5.991 which cuts off 5 percent of the right tail of the χ^2 distribution. In the smaller samples of both monthly and annual data, the overidentifying restrictions could not be rejected at the 5 percent significance level for most of the γ values.

5. Conclusions

While generalized method of moments techniques have the attractive feature of allowing one to estimate parameters of nonlinear rational expectations asset pricing models, any confidence in the estimates obtained through this technique must be tempered by the knowledge of the existence of finite sample bias. Our results show that this bias, in particular, becomes substantial for small values of the coefficient of relative risk aversion. Furthermore, it appears that introducing nonlinear functions and higher orders of polynomials in the instrument sets does not help to lessen the degree of bias. This suggests that finite sample bias may be an inherent feature of this estimation technique, and hypothesis testing which relies on asymptotic testing may be highly misleading. Our results confirm the findings by Tauchen that longer lag lengths in the instruments tend to lead to sampling distributions becoming more and more concentrated around severely biased values.

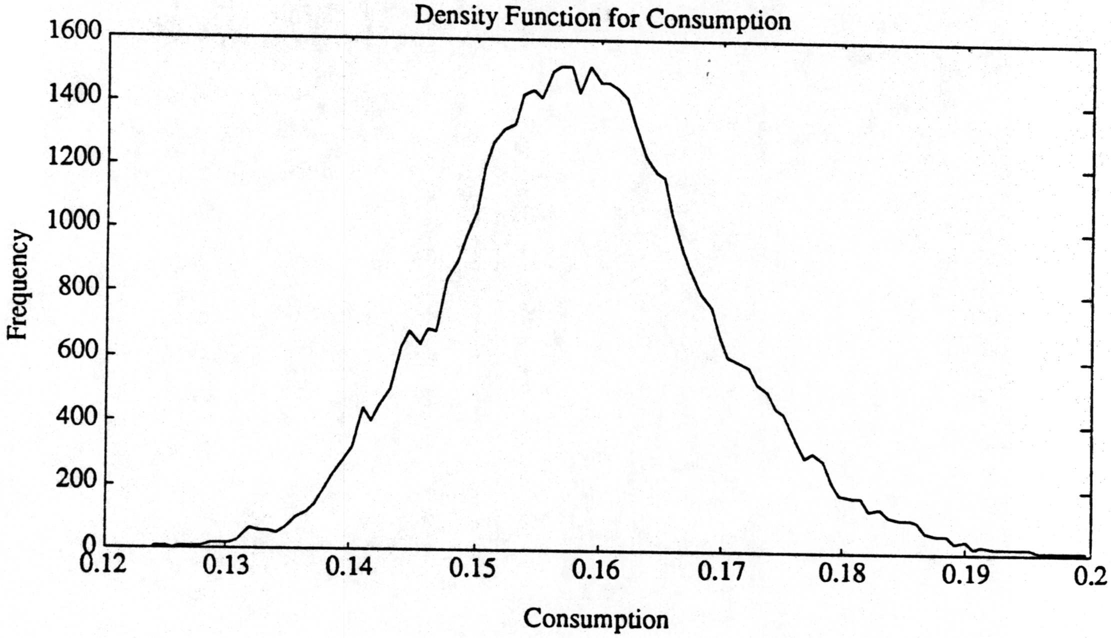
As Judd (1991) noted, economists are increasingly turning to numerical methods for analyzing dynamic economic models and this paper suggests that there are many avenues of research, previously only briefly explored, that such techniques will open up to researchers. Another possible area of work not unrelated to GMM estimation that this numerical technique may permit is to attempt to estimate and test the model with more than orthogonality conditions. Since one is able to solve for the decision rule for consumption, one can include other second moments such as the variance and covariance terms of the various processes. This may be of importance not only in discovering whether this inclusion eliminates the small sample bias problem but also in finding out whether these inclusions provide more efficient estimation of the parameters. This type of approach is in line with work by Duffie and Singleton (1989) and is an avenue for further research.

Summary Statistics on Simulated Time Series

Table 1

γ	m	TR^2	R^2	$\frac{var(k_{t+1}-k_t)}{var(\Delta c)}$
-0.5	70.26	9.79	0.42	28.95
-1.0	868.07	17.62	0.21	19.72
-1.5	45.49	5.60	0.06	9.45
-2.0	11.44	29.52	0.01	7.23
-2.5	510.33	14.42	0.02	0.88
-3.0	37.85	25.26	0.01	6.07
-3.5	18.60	14.12	0.02	6.77
-4.0	6.26	9.40	0.02	8.04
-4.5	108.32	14.56	0.01	4.60
-5.0	12.01	7.48	0.01	6.47
-5.5	106.48	7.42	0.03	7.38
-6.0	88.81	11.92	0.02	6.22
-6.5	30.51	21.90	0.01	5.78
-7.0	47.31	15.57	0.01	5.30
-7.5	25.04	13.58	0.01	3.80
-8.0	25.13	15.19	0.01	5.83
-8.5	41.64	14.44	0.01	3.91
-9.0	17.21	5.85	0.01	5.76
-9.5	21.60	22.17	0.01	5.51
-10.0	50.24	20.02	0.03	8.01
-10.5	32.14	9.68	0.01	5.97

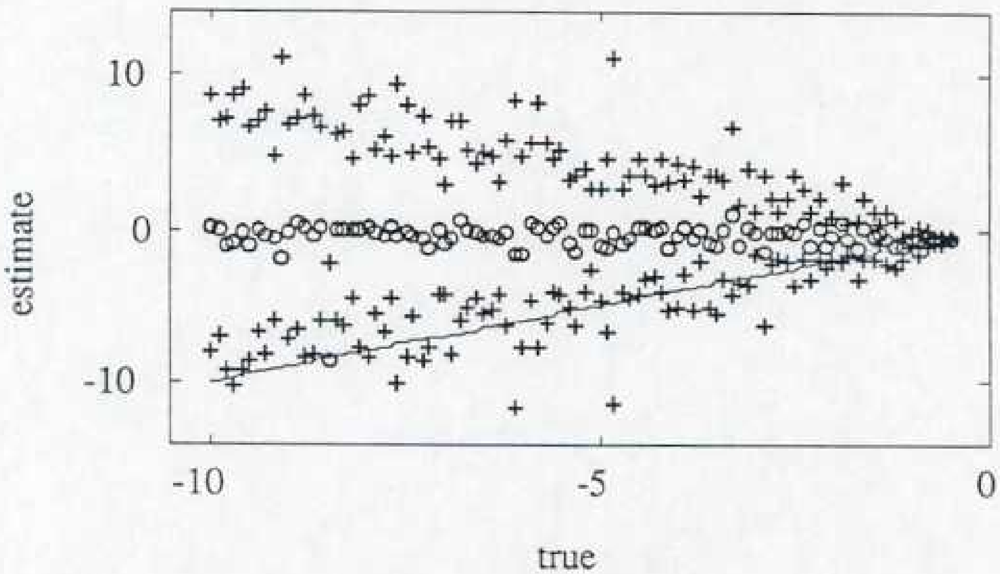
Figure 1



50000 months

Figure 2

GMM Estimates and Confidence Intervals



240 months, polynomial order 1, lag length 1

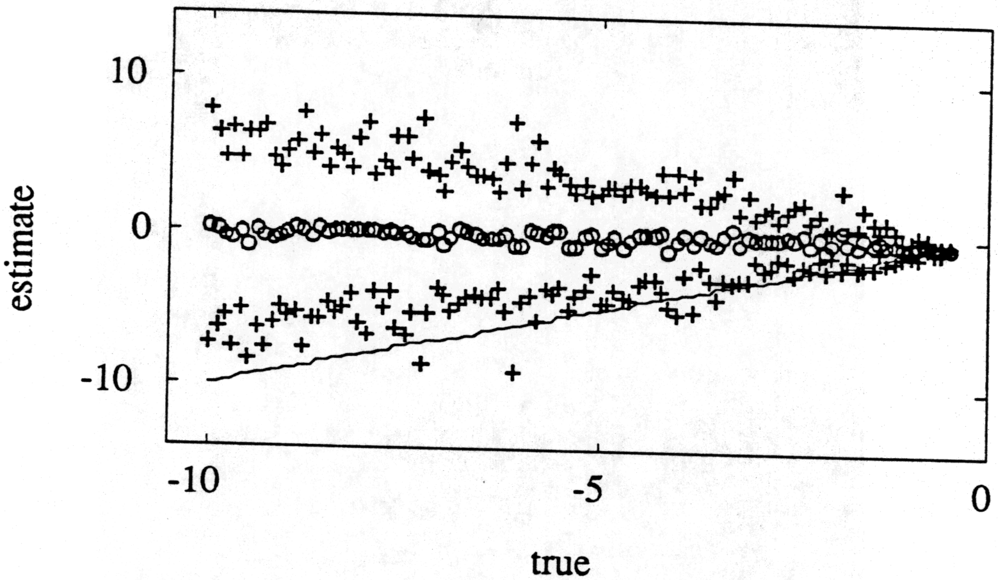
o γ estimate

$+$ confidence interval

$-$ true value of γ

Figure 3

GMM Estimates and Confidence Intervals



240 months, polynomial order 2, lag length 1

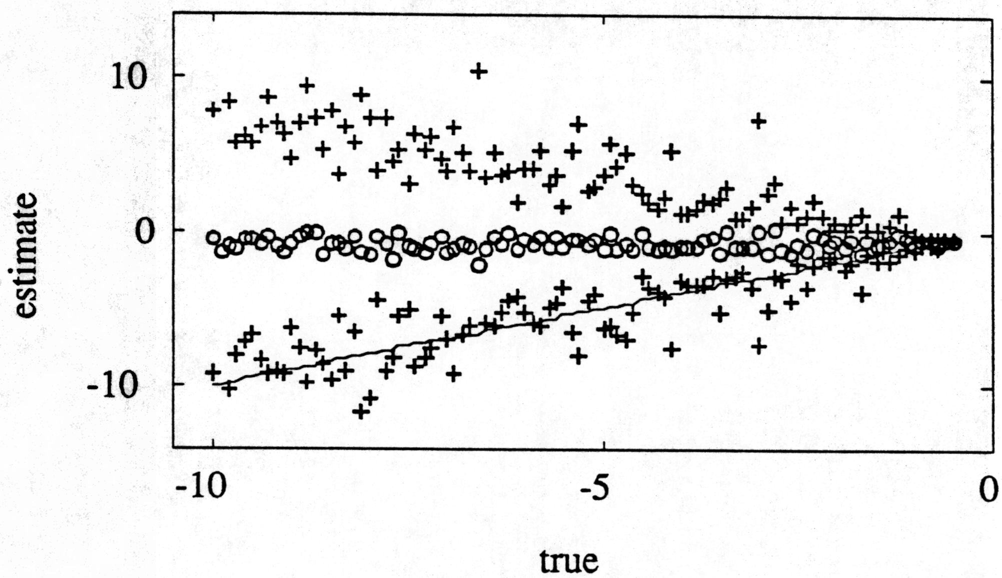
o γ estimate

+ confidence interval

− true value of γ

Figure 4

GMM Estimates and Confidence Intervals



40 years, polynomial order 1, lag length 1

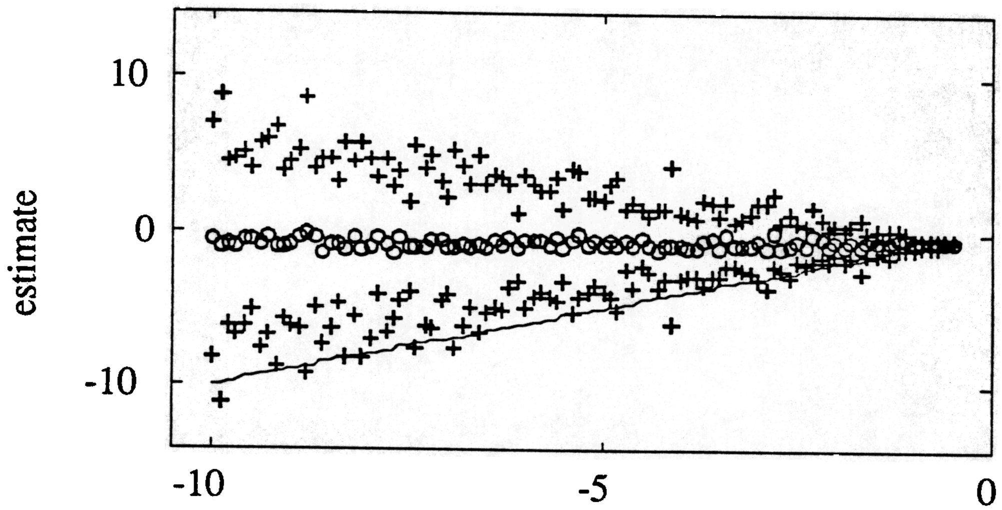
o γ estimate

+ confidence interval

– true value of γ

Figure 5

GMM Estimates and Confidence Intervals



40 years, polynomial order 2, lag length 1

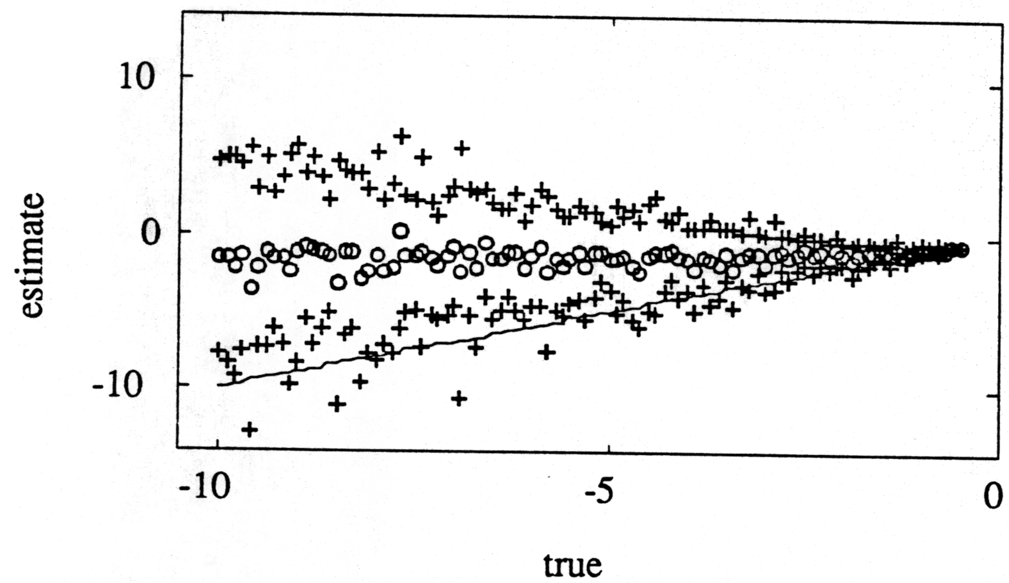
o γ estimate

\+ confidence interval

true value of γ

Figure 6

GMM Estimates and Confidence Intervals



240 years, polynomial order 2, lag length 1

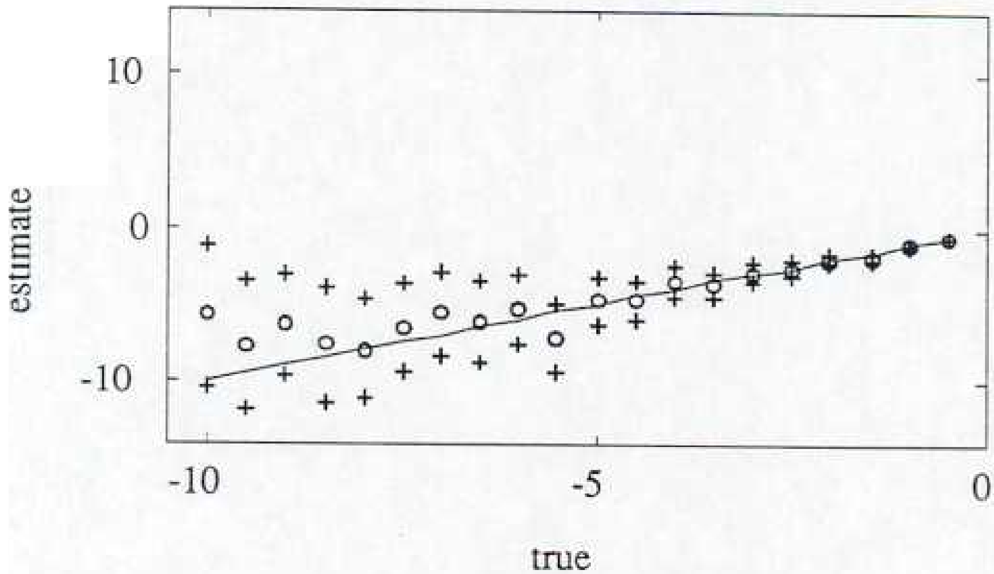
o γ estimate

+ confidence interval

– true value of γ

Figure 7

GMM Estimates and Confidence Intervals



24000 months, polynomial order 1, lag length 1

γ estimate

confidence interval

true value of γ

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